Applications of integrals

Business Mathematics

VU University Amsterdam
Faculty of Economics and Business Administration

CONTENTS

Areas under a curve

Consumer and producer surplus

From marginal to total

Old exam question

AREAS UNDER A CURVE

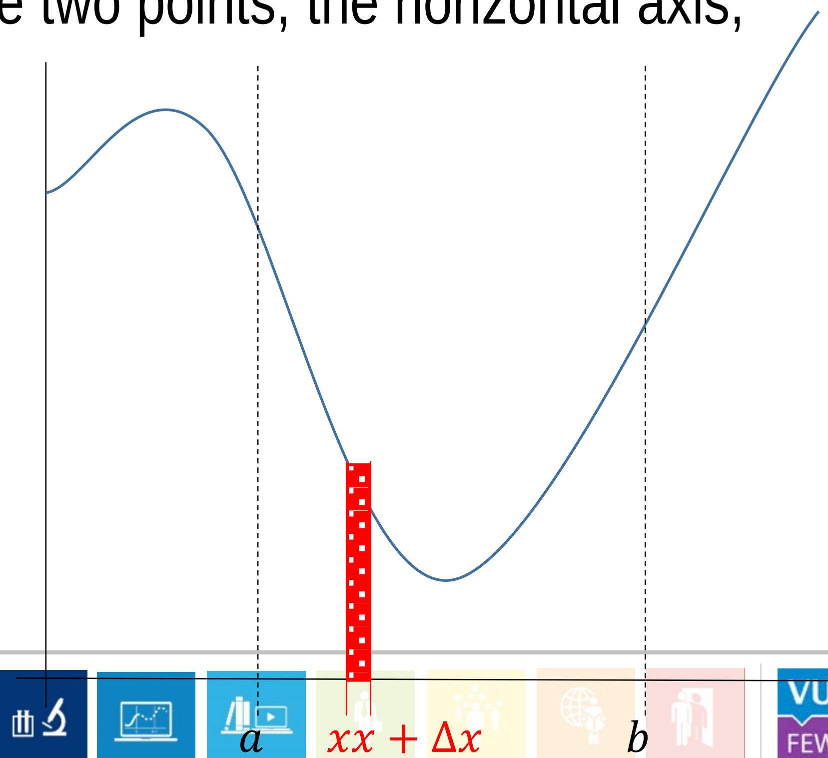
How do we compute the area A under the graph of a non-negative function f over the interval $[a, b]$?

Take a point x in the interval $[a, b]$

Move to a point slightly further, at $x + \Delta x$

Form a small rectangle enclosed by these two points, the horizontal axis, and the curve

- width Δx
- height $f(x)$
- area $\Delta A = f(x)\Delta x$



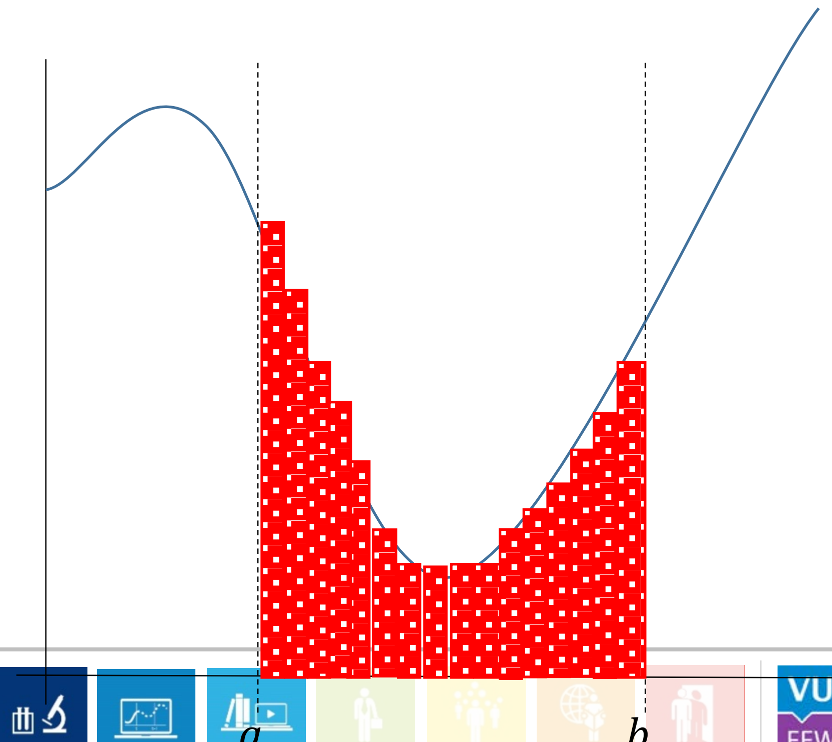
AREAS UNDER A CURVE

Let x run from a to b in n steps

- clearly $\Delta x = \frac{b-a}{n}$
- each rectangle starts at $x_i = a + (i-1)\Delta x$

Then $A = \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} f(x_i)\Delta x$

- Riemann sum



AREAS UNDER A CURVE

Now take Δx very small (n very big)

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

The Riemann sum provides one way of proving that the area under a non-negative function $f(x)$ within in an interval $[a, b]$ (or (a, b)) is $\int_a^b f(x)dx$

VU FEWEB

AREAS UNDER A CURVE

Another way:

- suppose $A(x)$ is the area between a and x.
- consider $x+\Delta x$, the area is $A(x+\Delta x) \approx A(x) + f(x)\Delta x$
- so $\Delta A(x) \approx f(x)\Delta x$ and $\dfrac{\Delta A(x)}{\Delta x} \approx f(x)$
- so in the limit $\lim\limits_{\Delta x \to 0} \dfrac{\Delta A(x)}{\Delta x} = \dfrac{dA(x)}{dx} = f(x)$

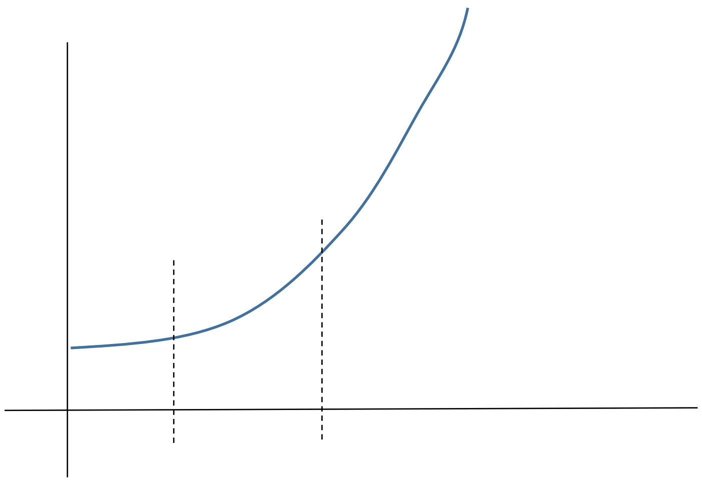
In words:

- the derivative of the area function A is the function f
- so the area function A is a primitive function of the function f

AREAS UNDER A CURVE

Example: let $f(x) = x^2 + 5$, with integration boundaries $[1,2]$

- check that $f(x) \geq 0$ within $[1,2]$



The area enclosed by:

- the x-axis ($y = 0$)
- the function ($y = f(x)$)
- the lower boundary ($x = 1$)
- the upper boundary ($x = 2$)

... is $A = \int_1^2 f(x)dx$

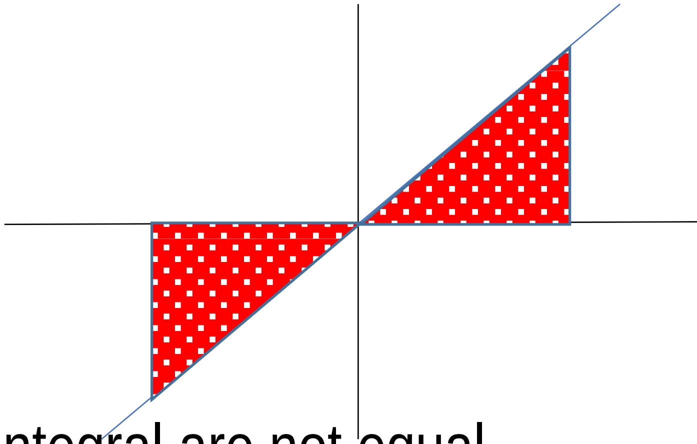
Because $\int f(x)dx = \int (x^2 + 5)dx = \frac{1}{3}x^3 + 5x + C$,

we have $\mathrm{A} = \int_1^2 f(x)dx = \left[\frac{1}{3}x^3 + 5x\right]_1^2 = \frac{1}{3}2^3 + 5 \cdot 2 - \left(\frac{1}{3}1^3 + 5 \cdot 1\right) = \frac{8}{3} + 10 - \frac{1}{3} - 5 = 7\frac{1}{3}$

AREAS UNDER A CURVE

Let's compute $\int_{-1}^{1} x dx = \frac{1}{2}x^2\Big]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

Note that 0 is not the area



If $f(x)$ takes a negative value, area and definite integral are not equal

In order to compute the area one proceeds as follows:

- $A = \int_{-1}^{0}(-x)dx + \int_{0}^{1} x dx = \frac{1}{2}x^2\Big]_{-1}^{0} - \frac{1}{2}x^2\Big]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$

CONSUMER AND PRODUCER SURPLUS

Let $f(Q)$ denote the demand curve

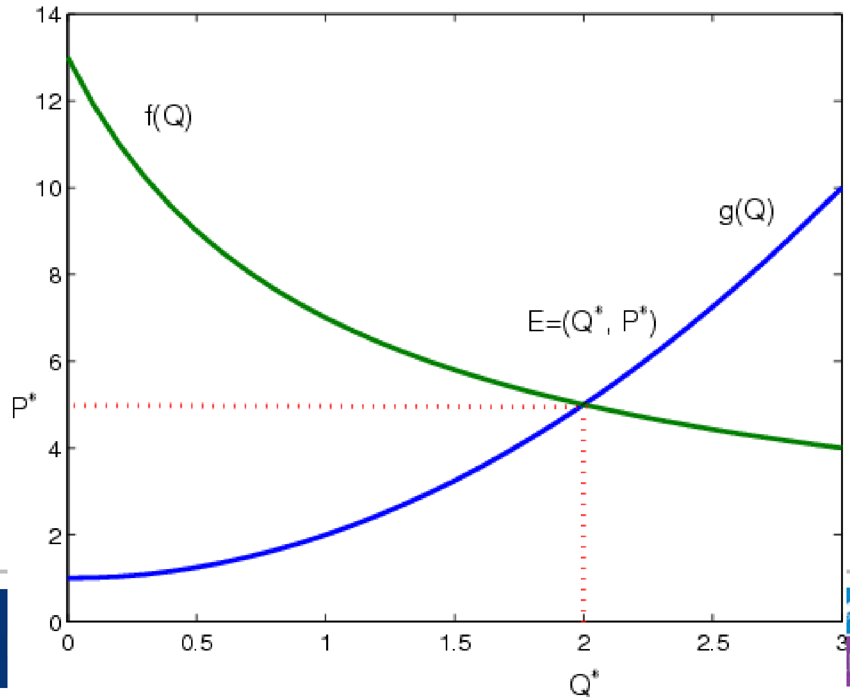
- consumers will buy quantity Q if the price is $f(Q)$

Let $g(Q)$ denote the supply curve

- producers will produce quantity Q if the price is $g(Q)$

Let $E = (Q^*, P^*)$ be the equilibrium point

- where $f(Q^*) = P^* = g(Q^*)$



CONSUMER AND PRODUCER SURPLUS

Define the consumer surplus

$$CS = \int_0^{Q^*} (f(Q) - P^*)dQ$$

and the producer surplus

$$PS = \int_0^{Q^*} (P^* - g(Q))dQ$$

CONSUMER AND PRODUCER SURPLUS

Example:

- $f(Q) = 1 + \frac{12}{Q+1}$
- $g(Q) = Q^2 + 1$

Equilibrium point:

- $1 + \frac{12}{Q^*+1} = {Q^*}^2 + 1 \Rightarrow {Q^*}^2(Q^* + 1) = 12 \Rightarrow$
- $(Q^*, P^*) = (2{,}5)$

Surplus:

- $CS = \int_0^2 \left(\frac{12}{Q+1} - 4\right) dQ = (12\ln(Q+1) - 4Q)]_0^2 = 12\ln 3 - 8 \approx 5.18$
- $PS = \int_0^2 (4 - Q^2) dQ = \left(4Q - \frac{Q^3}{3}\right)\Big]_0^2 = \frac{16}{3} \approx 5.33$

FROM MARGINAL TO TOTAL

Given a cost function, we can calculate the marginal cost with a derivative

- example: $C(q) = 20 + 4q^{0.6}$
- marginal costs: $C'(q) = 2.4q^{-0.4}$

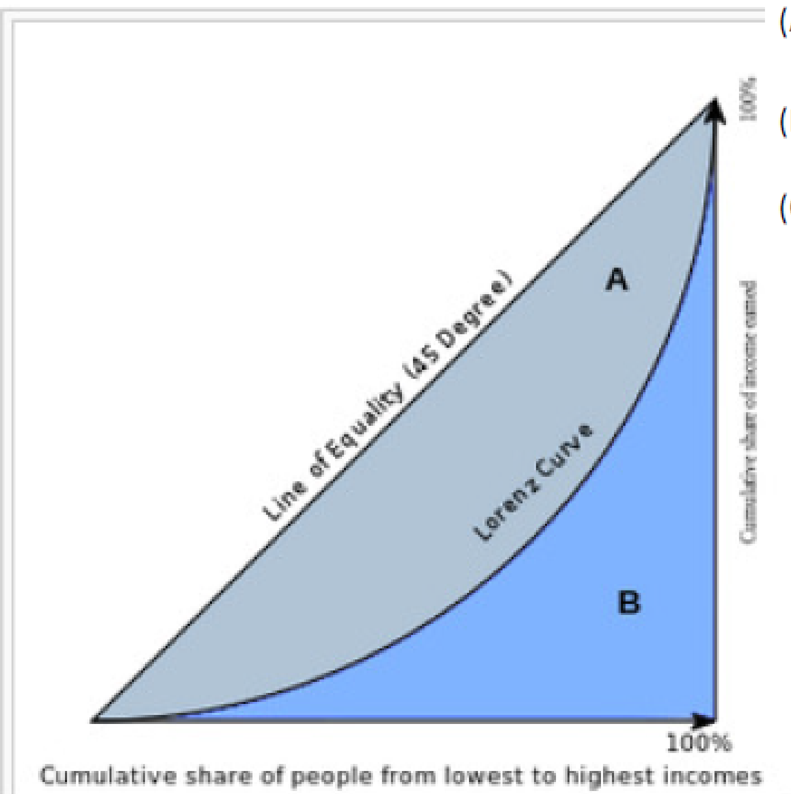
But suppose we know the marginal costs and need the cost function

- example: $C'(q) = 2.4q^{-0.4}$
- cost function: $C(q) = \int C'(q)dq = 4q^{0.6} + C$
- unique up to the integration constant C

OLD EXAM QUESTION

27 March 2015, Q1i

The Gini coefficient is an index of income inequality. Wikipedia illustrates it with the graph below. Let the line of equality be given by $y = x$ and the curved line by $y = f(x)$. Then the Gini coefficient equals the area A, given by ...... (choose one)



(A) $A = \int_0^{100\%} (xg(x))dx$

(B) $A = \int_0^{100\%} (x + g(x))dx$

(C) $A = \int_0^{x} g(x)dx$

(D) $A = \int_0^{100\%} \left(\frac{x}{g(x)}\right) dx$

(E) $A = \int_0^{100\%} (x - g(x))dx$

(F) $A = \int_0^{y} g(x)dx$

(G) None of the above is correct.