Constrained optimization

Business Mathematics

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UNCONSTRAINED OPTIMIZATION

Recall extreme values in two dimensions

- first-order conditions: stationary points occur when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$
- second-order conditions: extreme value when in addition $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$
- nature of extreme value:
  - minimum when $\frac{\partial^2 f}{\partial x^2} > 0$
  - maximum when $\frac{\partial^2 f}{\partial x^2} < 0$

UNCONSTRAINED OPTIMIZATION

Recall Cobb-Douglas production function

- $q(K, L) = AK^{\alpha}L^{\beta}$

Obviously (?) you want to maximize output

- so set $\frac{\partial q}{\partial K} = A\alpha K^{\alpha-1}L^{\beta} \ = 0$
- and $\frac{\partial q}{\partial L} = AK^{\alpha}\beta L^{\beta-1} \ = 0$

This never happens!

But you still want to maximize output

- within the constraints of your budget

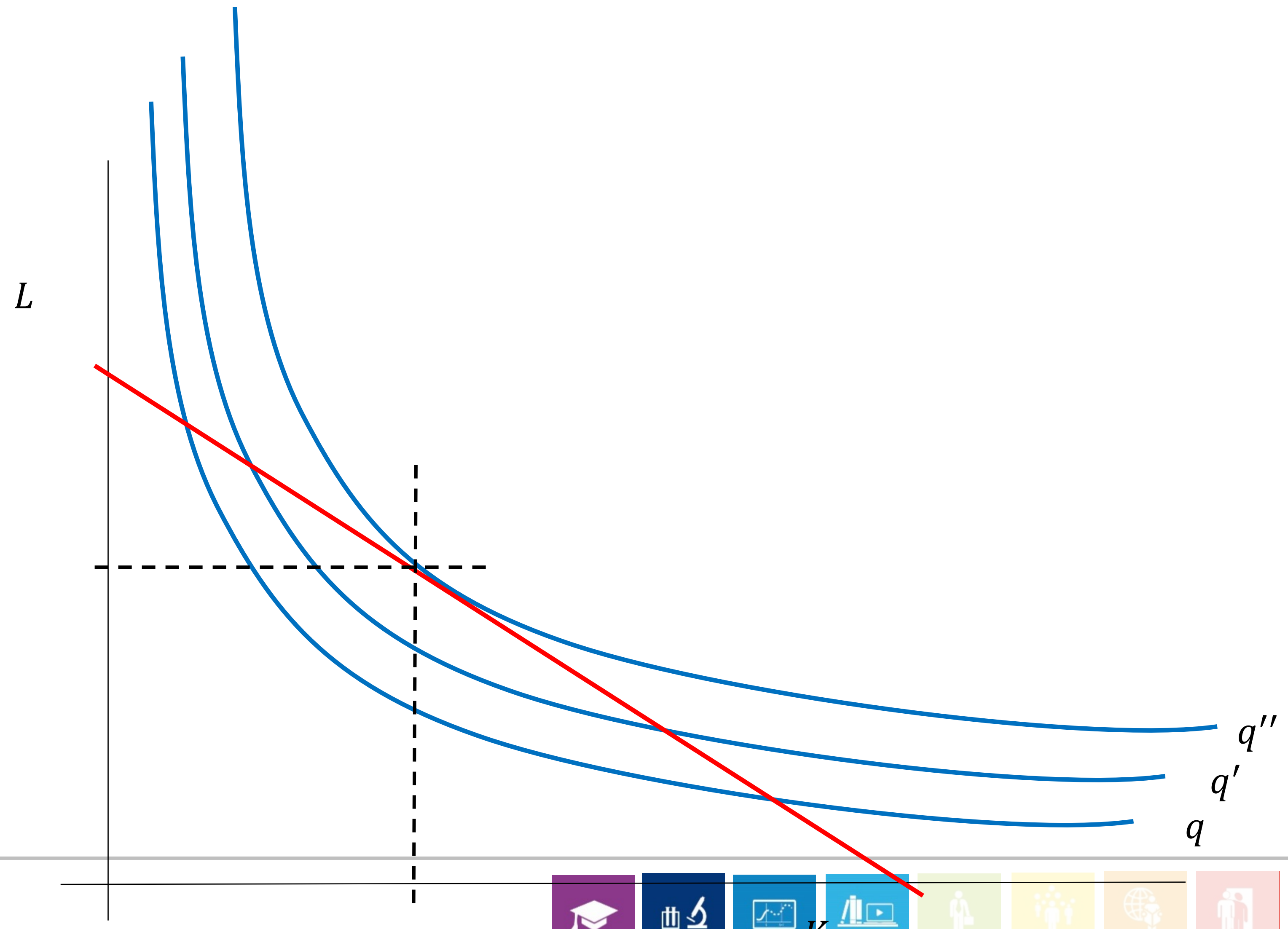
CONSTRAINED OPTIMIZATION

Suppose:

- the price of 1 unit of K is k
- the price of 1 unit of L is l
- the available budget is m

Budget line: $kK + lL = m$

CONSTRAINED OPTIMIZATION



CONSTRAINED OPTIMIZATION

Problem formulation:

- $$\begin{cases} \text{maximize} & q(K,L) = AK^{\alpha}L^{\beta} \\ \text{subject to} & kK + lL = m \end{cases}$$

More in general

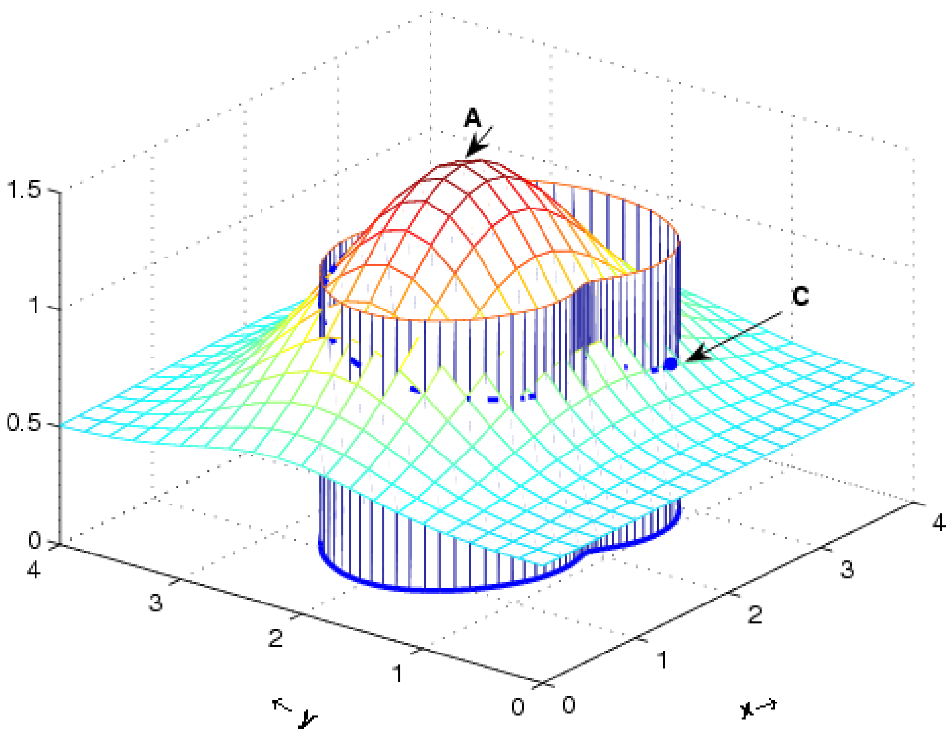
$$\boxed{\begin{cases} \max & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{cases}}$$

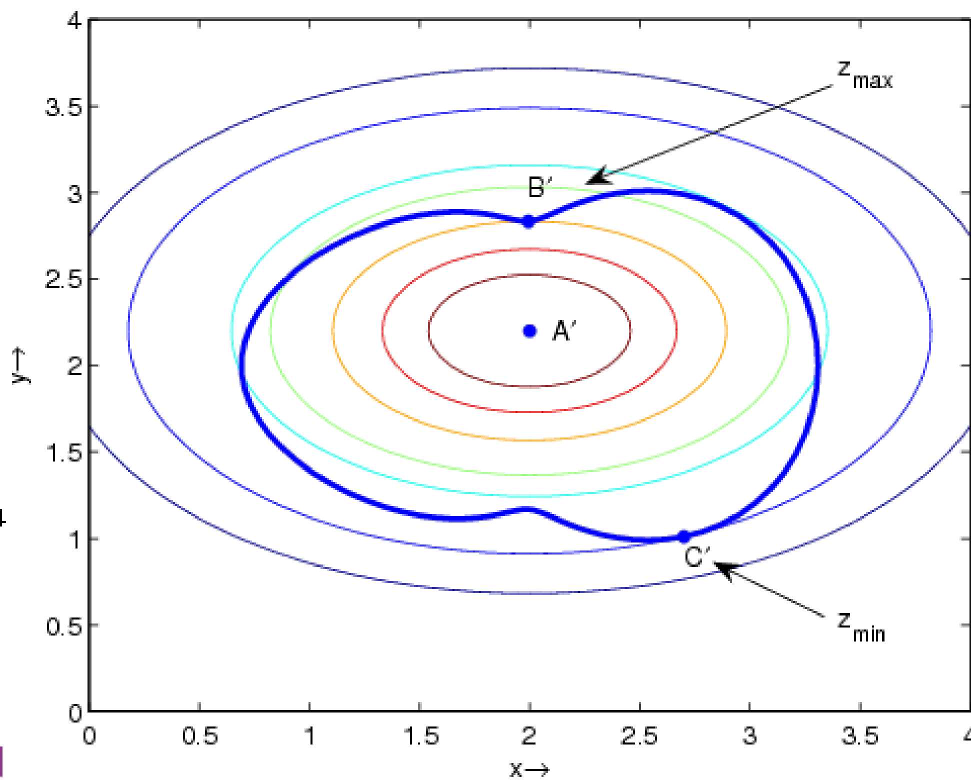
Constrained optimization problem

CONSTRAINED OPTIMIZATION

Visualisation of $z = f(x, y)$

- in 3D
- as level curve





LAGRANGE METHOD

How to solve the constrained optimization problem?

$$\begin{cases} \max & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{cases}$$

Trick:

- introduce extra variable (Lagrange multiplier) λ
- define new function (Lagrangian) $\mathcal{L}(x,y,\lambda)$
- as follows

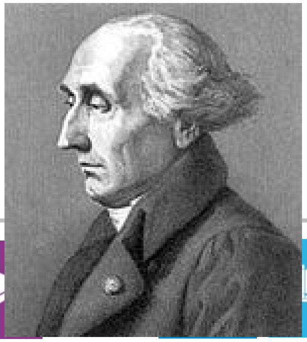
$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$$

Motivation:

- solutions of the constrained optimization are among the stationary points of $\mathcal{L}$

Lagrange method

- Joseph-Louis Lagrange (1736-1813)



LAGRANGE METHOD

Example 1

Constrained problem:

- $\begin{cases} \max & f(x,y) = x^2 + y^2 \\ \text{s.t.} & g(x,y) = x^2 + xy + y^2 = 3 \end{cases}$

Lagrangian:

- $\mathcal{L}(x,y,\lambda) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$

First-order conditions for stationary points:

- $\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda(2x + y) = 0$
- $\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda(x + 2y) = 0$
- $\frac{\partial \mathcal{L}}{\partial \lambda} = -x^2 - xy - y^2 + 3 = 0$

LAGRANGE METHOD

Four stationary points:

- $(x, y, \lambda) = \left(1, 1, \frac{2}{3}\right)$
- $(x, y, \lambda) = \left(-1, -1, \frac{2}{3}\right)$
- $(x, y, \lambda) = \left(\sqrt{3}, -\sqrt{3}, 2\right)$
- $(x, y, \lambda) = \left(-\sqrt{3}, \sqrt{3}, 2\right)$

Determine nature of these points

- use common sense

LAGRANGE METHOD

Example 2

Constrained problem:

- $\begin{cases} \text{maximize} & q(K,L) = AK^{0.4}L^{0.7} \quad (A > 0) \\ \text{subject to} & 25K + 10L = m \quad (m > 0) \end{cases}$

Lagrangian

- $\mathcal{L}(K,L,\lambda) = AK^{0.4}L^{0.7} - \lambda(25K + 10L - m)$

First-order conditions for stationary points

- $\dfrac{\partial \mathcal{L}}{\partial K} = 0.4AK^{-0.6}L^{0.7} - 25\lambda = 0$
- $\dfrac{\partial \mathcal{L}}{\partial L} = 0.7AK^{0.4}L^{-0.3} - 10\lambda = 0$
- $\dfrac{\partial \mathcal{L}}{\partial \lambda} = -(25K + 10L - m) = 0$

LAGRANGE METHOD

Example 2 (continued)

Solution:

- $K = \frac{0.7}{0.7+0.4}\frac{m}{10}, L = \frac{0.4}{0.7+0.4}\frac{m}{25}, \lambda = \cdots$ is only stationary point

LAGRANGE METHOD

Example 2 (continued)

Determine nature of the stationary point:

- take $K = 0 \Rightarrow L = \frac{m}{10}$ and see that $q = 0$
- so for $K = \frac{0.7}{0.7+0.4}\frac{m}{10} > 0$ and $L = \frac{0.4}{0.7+0.4}\frac{m}{25} > 0$, also $q > 0$
- therefore $\left(\frac{0.7}{0.7+0.4}\frac{m}{10}, \frac{0.4}{0.7+0.4}\frac{m}{25}\right)$ is a maximum

Here, we look at two ”neighbour” points of the stationary point: one “to the left” and one “to the right”. We find that both have a lower function value, so the stationary point is a maximum.

OLD EXAM QUESTION

22 October 2014, Q2a

A manufacturer produces and sells two products in amounts x_1 and x_2. It has a profit function $\pi(x_1, x_2) = 4x_1x_2$ and a budget function $C(x_1, x_2) = x_1^2 + 4x_2^2$ (of which the level curves have the shape of an ellipse around the origin). Find the sale levels of x_1 and x_2, such that profit is maximized within a budget constraint of 32. Check the character of the stationary point (maximum or minimum) by using common sense (13 points)

OLD EXAM QUESTION

27 March 2015, Q2b'

A busy salesman has exactly 30 minutes to do exercises. He dislikes most machines but only uses machines 1 and 2, during a time period t_1 and t_2. His doctor has told him that his overall fitness (f) is given by $f = (t_1)^{0.8}(t_2)^{0.4}$. Use Lagrange's method to find how the salesman should divide his 30 minutes of exercise over the two machines to obtain maximum health. (12 points)