Derivatives

Business Mathematics

VU UNIVERSITY AMSTERDAM
Faculty of Economics and Business Administration

CONTENTS

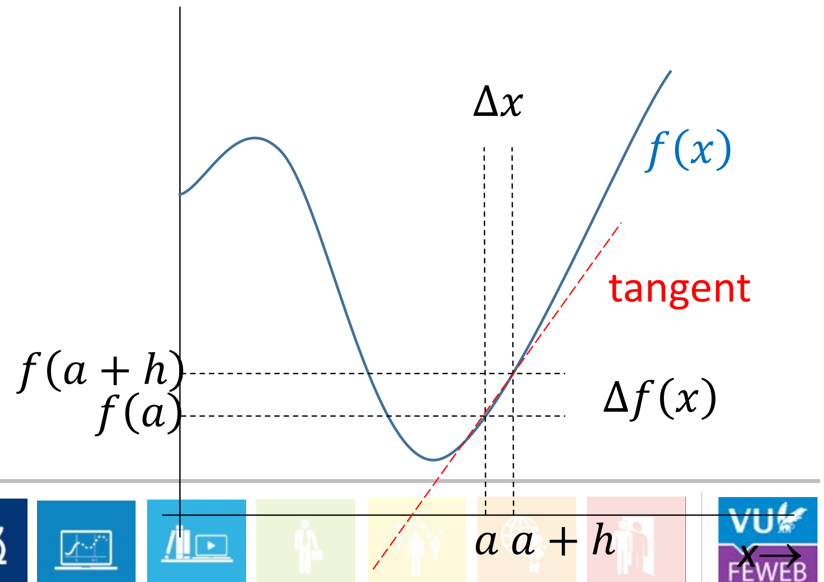
TANGENTS TO AND SLOPES OF CURVES

Let $f(x)$ be a smooth function on a domain D

- at any point $x = a$ within D the function $f(x)$ has a tangent
- its slope can be approximated as

$$\left.\frac{\Delta f(x)}{\Delta x}\right]_{x=a} \approx \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

the slope at a point $(a, f(a))$ is a number between $-\infty$ and ∞



TANGENTS TO AND SLOPES OF CURVES

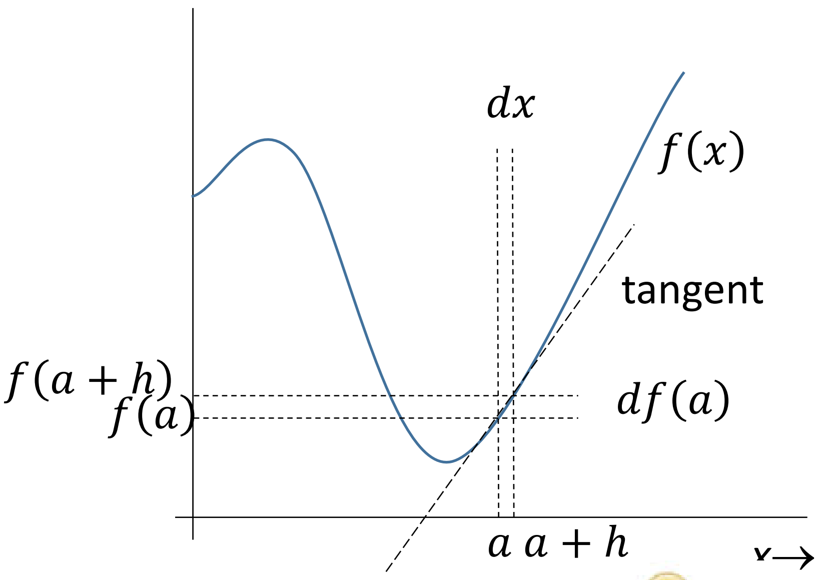
When h approaches zero, the slope of the tangent of $f(x)$ at a is given by

$$\left.\frac{df(x)}{dx}\right]_{x=a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Or more general: the derivative

$$\boxed{\frac{df(x)}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

the derivative of a function $f(x)$ depends on x; it is also a function



FEWEB

TANGENTS TO AND SLOPES OF CURVES

Different notations:

- $\frac{df(a)}{dx}$
- $\left.\frac{df}{dx}\right]_a$
- $f'(a)$
- $\dot{f}(a)$
- $\frac{d}{dx}f(a)$
- etc.

DERIVATIVES

The derivative $f'(a)$ of a function $f(x)$ at a specified place a is a number

- if $f(x) = 4x^3$, then $f'(2) = 48$
- also written as $\left[\frac{df(x)}{dx}\right]_{x=2} = 48$

The derivative $f'(x)$ of a function $f(x)$ over a range of values x is a function of x

- if $f(x) = 4x^3$, then $f'(x) = 12x^2$

BASIC CASES FOR DERIVATIVES

Constant function:

- if $f(x) = C$, then $f'(x) = 0$

Linear function:

- if $f(x) = ax$, then $f'(x) = a$

Power function:

  - if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ (for arbitrary n)

Exponential function:

- if $f(x) = e^x$, then $f'(x) = e^x$

Logarithmic function:

- if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

BASIC CASES FOR DERIVATIVES

Each of these can be proven (or seen for a special case)

- example: $f(x) = x^3$:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2$$

And similarly (but sometimes with more difficulty) for the other cases

RULES FOR DERIVATIVES

Four main rules:

- Sum rule: $(f+g)' = f' + g'$
- Product rule: $(fg)' = f'g + fg'$
- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Chain rule: $g\big(f(x)\big)' = g'\big(f(x)\big)f'(x)$

Not restricted to f and x

- if $q(\lambda) = 4\lambda^3$, then $q'(\lambda) = 12\lambda^2$
- or $\frac{dq}{d\lambda} = 12\lambda^2$

RULES FOR DERIVATIVES

Three examples:

- Example 1:
  $$f(x) = x^3 - 10x^2 + 16x + 55 \Rightarrow f'(x) = 3x^2 - 20x + 16$$
- Example 2:
  $$f(x) = \frac{1+x^2}{x} \Rightarrow f'(x) = \left(\frac{1+x^2}{x}\right)'$$
  $$= \frac{(1+x^2)'x - (1+x^2)x'}{x^2} = \frac{(0+2x)x - (1+x^2)1}{x^2}$$
  $$= \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$$
- Example 3:
  $$f(x) = a^x \Rightarrow f'(x) = \left(e^{x \ln a}\right)' = (\ln a)e^{x \ln a} = (\ln a)a^x$$

HIGHER-ORDER DERIVATIVES

The derivative $f'(x)$ is a function of x

It can be subject to differentiation again

- $(f'(x))' = f''(x)$
- $\frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2 f(x)}{dx^2} = \frac{d^2}{dx^2}\left(f(x)\right)$
- second derivative, or second-order derivative

Example:

- $f(x) = 3x^5 + x$
- $\Rightarrow f'(x) = 15x^4 + 1$
- $\Rightarrow f''(x) = 60x^3$

And similar, third-order, n^{th}-order

HIGHER-ORDER DERIVATIVES

To find the second derivative at a point $x = a$, first find the first derivative as a function

Example

- let $f(x) = 3x^5 + x$, find $f''(2)$
- first calculate $f'(x) = 15x^4 + 1$
- then do not calculate $f'(2)$
- but rather calculate $f''(x) = 60x^3$
- finally calculate $f''(2) = 60 \times 8 = 480$

Again, the second derivative is a function of x, but evaluated at a point $x = a$, it is a number

MORE ON DERIVATIVES

Extensions:

- functions of more variables (partial derivatives)
- implicit functions (implicit differentiation)

Applications:

- extreme values (minimum costs, maximum profits, etc.)
- elasticities (price elasticity of demand, etc.)
- approximations (Taylor series)

OLD EXAM QUESTION

23 April 2014, Q1m

Given is $f(x) = \sqrt{\sqrt{x}}$. Evaluate $f'(16)$. (exact)

FEWEB

OLD EXAM QUESTION

23 April 2015, Q2a

A business analyst has calculated that nation-wide, the costs C (in euro) of a failure are related to the duration of the interrupt t (in minutes) as $C = a(e^{bt} - 1)$, where a and b are constants. We only consider $t \geq 0$. This model should satisfy the following conditions:

1) it should not predict negative costs (so $C \geq 0$)

2) the costs should increase with increasing t (so $\frac{dC}{dt} > 0$)

3) the costs should increase with increasing t at an increasing speed (so $\frac{d^2C}{dt^2} > 0$)

For which values of a and b are these three conditions satisfied? (5 points)