Extreme values

Business Mathematics

VU UNIVERSITY AMSTERDAM
Faculty of Economics and Business Administration

CONTENTS

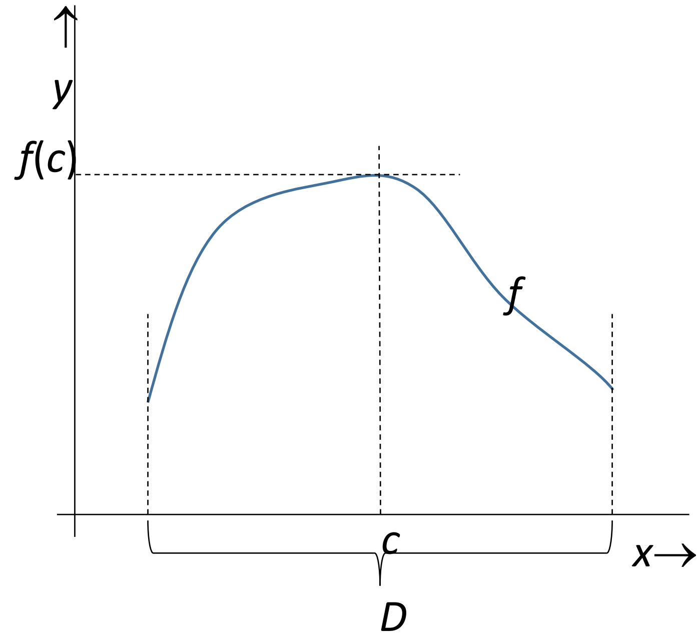
EXTREME POINTS AND VALUES

Let $f(x)$ have domain D

We call $c \in D$ a maximum point for f if and only if $f(x) \leq f(c)$ for all $x \in D$

We call $f(c)$ a maximum value of f

If $f(c) > f(x)$ for all $x \in D$ with $x \neq c$, then c is a strict maximum



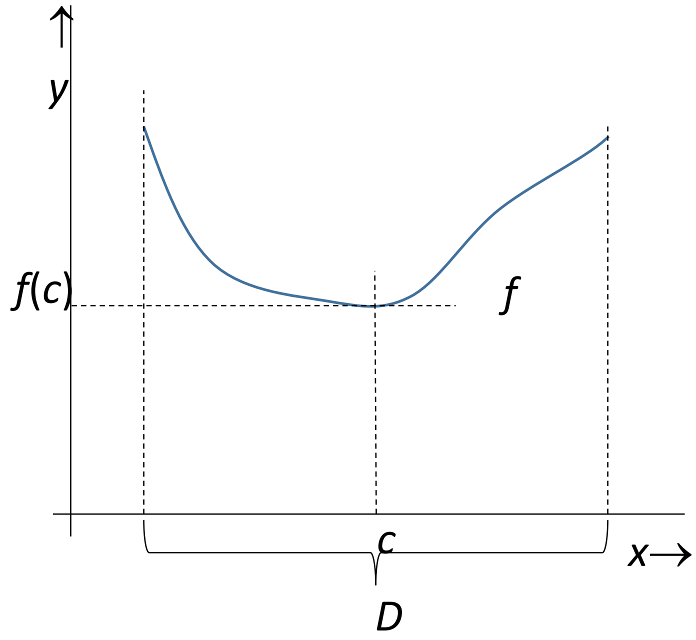
EXTREME POINTS AND VALUES

Similar for minimum points and minimum values

Collectively denoted as optimal points and optimal values, or extreme points and extreme values

Important in business and economics:

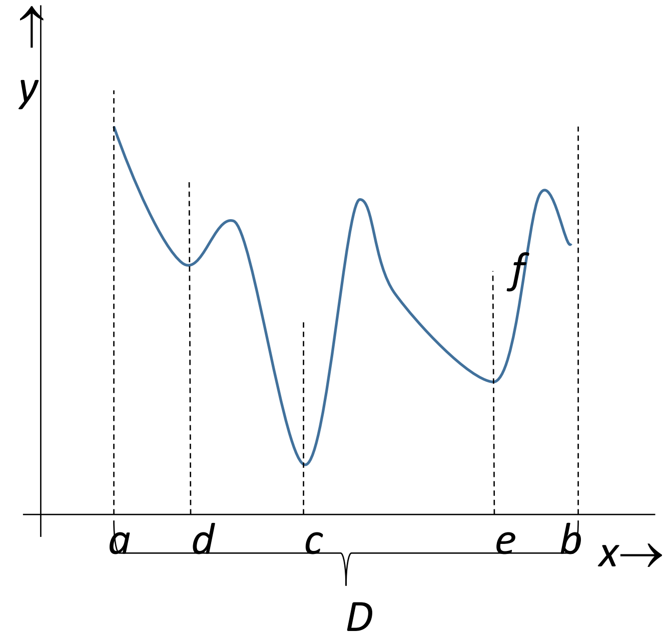
- maximum profit, maximum sales, etc.
- minimum costs, minimum risk, etc.



EXTREME POINTS AND VALUES

Local and global extreme points

- local: if the extreme point is an extreme in a small interval within D
- global: if the extreme point is an extreme over the full domain D



Example:

- global minimum point at c
- local minimum points at c, d, e, and the upper boundary of D (b)

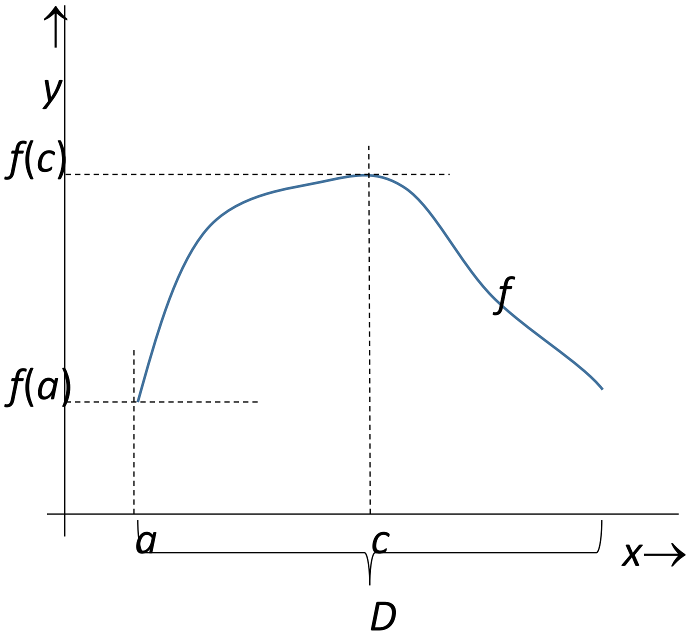
Example:

- global maximum point at a
- also several local maximum points

EXTREME POINTS AND VALUES

For smooth functions, extreme values may occur

- at points (like at c) where $f' = 0$ (stationary points)
- at the boundary of the domain D (like at a)



Points not at the boundary are interior points of the domain D

FIRST-ORDER CONDITIONS

Necessary condition for an extreme value at an interior point at $x = c$

$$\boxed{f'(c) = 0}$$

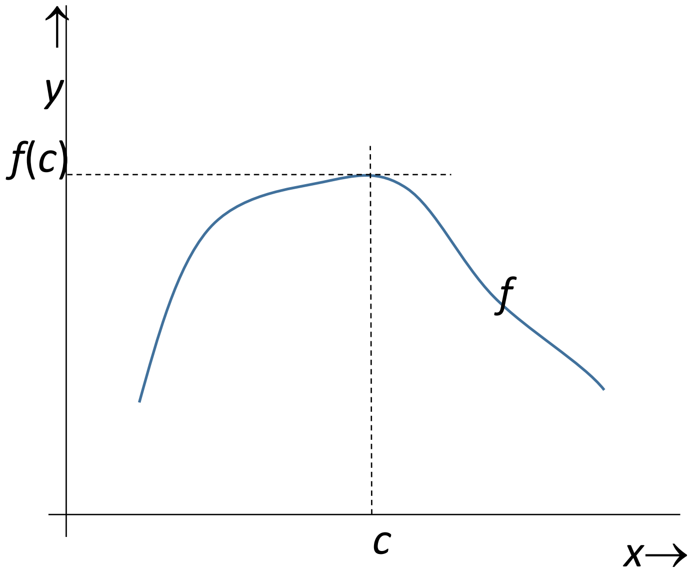
But not a sufficient condition

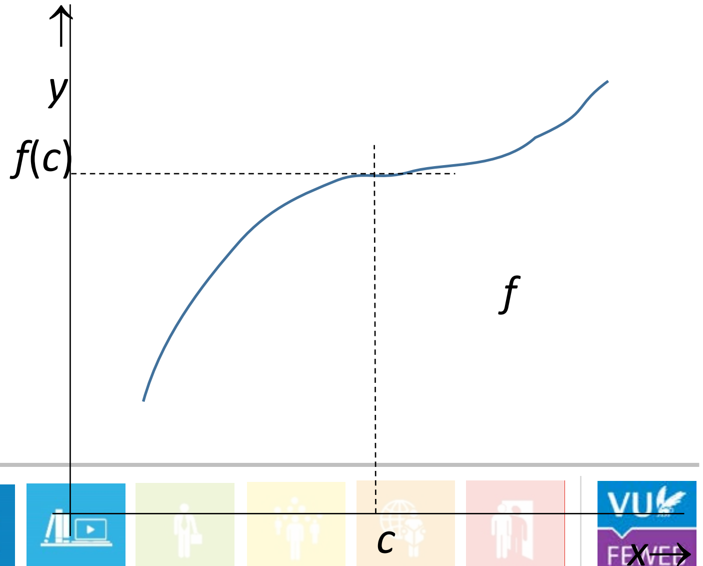
Possible stationary points:

- minimum
- maximum
- <u>saddle point/inflection point</u>

<u>First-order condition</u>

- first-order derivative
- a necessary condition





FIRST-ORDER CONDITIONS

Example

- consider $f(x) = 1 - \frac{2}{2+x^2}$
- differentiating yields $f'(x) = \frac{4x}{(2+x^2)^2}$
- solving $f'(x) = 0$ yields $x = 0$
- the function has only one stationary point at $x = 0$

SECOND-ORDER CONDITIONS

Let f be a twice differentiable function. Suppose that c is a stationary point, then:

- if $f''(c) < 0$, then c is a local maximum point
- if $f''(c) > 0$, then c is a local minimum point
- if $f''(c) = 0$, then further investigation is required

Second-order condition

- second-order derivative
- $f'' \neq 0$ is a necessary condition for an extreme value

SECOND-ORDER CONDITIONS

Example

- consider again $f(x) = 1 - \frac{2}{2+x^2}$ with $f'(x) = \frac{4x}{(2+x^2)^2}$
- what is the type of the stationary point at $x = 0$?
- differentiating $f'(x)$ yields $f''(x) = -\frac{4\left(3x^2-2\right)}{(2+x^2)^3}$
- inserting $x = 0$ yields $f''(0) = 1 > 0$
- so $x = 0$ is a local minimum

FINDING EXTREME VALUES

1. Check domain and differentiability of f
   - find values of f at the boundaries of the domain and at points where f is non-differentiable
2. Find all stationary points with $f' = 0$
3. Check nature of each stationary point
   - calculate the values of f at each local extreme point

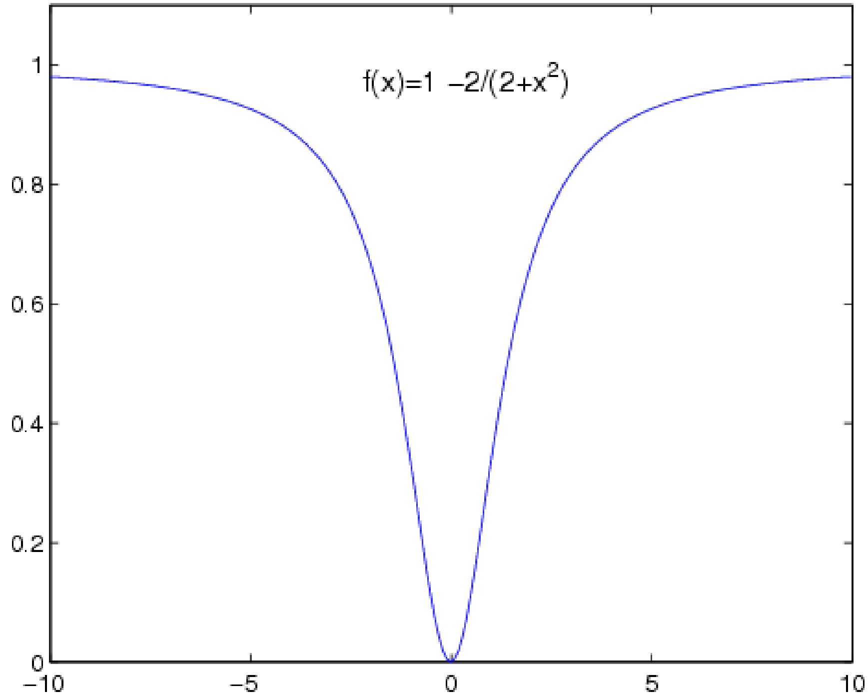
Local extreme values: ready

Global extreme values: find which ones classify as global

FINDING EXTREME VALUES

Example

- f is defined and differentiable for all x
- $f(x)$ has a local minimum at $x = 0$ and there is no other local minimum
- from the graph of the function we conclude that $x = 0$ is the strict minimum of $f(x)$
- note that $f(x)$ has no point of local or strict maximum even though $f(x)$ approaches 1 from below
- sign diagram:



$f''(x)$	$-$
$f'(x)$	$- - - -0 + + + +$
$f(x)$	min

OLD EXAM QUESTION

10 December 2014, Q1c

A function $f(x)$ that is defined on domain $x \in [a, b]$ is known to have a derivative $f'(x) > 0$ on that domain. Which statement(s) is (are) correct? (choose one or more)

(A) $f(x)$ has an interior maximum.

(B) $f(x)$ has a maximum at $x = a$

(C) $f(x)$ has a maximum at $x = b$.

(D) $f(x)$ has an interior minimum.

(E) $f(x)$ has a minimum at $x = a$.

(F) $f(x)$ has a minimum at $x = b$.

(G) None of the answers above is correct.

OLD EXAM QUESTION

27 March 2015, Q1e

Determine both coordinates $(x, g(x))$ of the minimum point of the function $g(x) = e^{(2x-3)^2}$.
(exact)