Extreme values in two dimensions

Business Mathematics

VU University Amsterdam
Faculty of Economics and Business Administration

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EXTREME VALUES IN ONE DIMENSION

Consider a smooth function $f(x)$

Necessary first-order condition for extreme value

- $\frac{df(x)}{dx} = 0 \Rightarrow \text{stationary point}$

Additional sufficient second-order test

- $\frac{d^2 f(x)}{dx^2} < 0 \Rightarrow \text{maximum}$
- $\frac{d^2 f(x)}{dx^2} > 0 \Rightarrow \text{minimum}$
- $\frac{d^2 f(x)}{dx^2} = 0 \Rightarrow ?$

EXTREME VALUES IN TWO DIMENSIONS

Consider a smooth function $f(x, y)$

Necessary first-order condition for extreme value

- $\frac{\partial f(x,y)}{\partial x} = 0$ and $\frac{\partial f(x,y)}{\partial y} = 0 \Rightarrow \text{stationary point}$

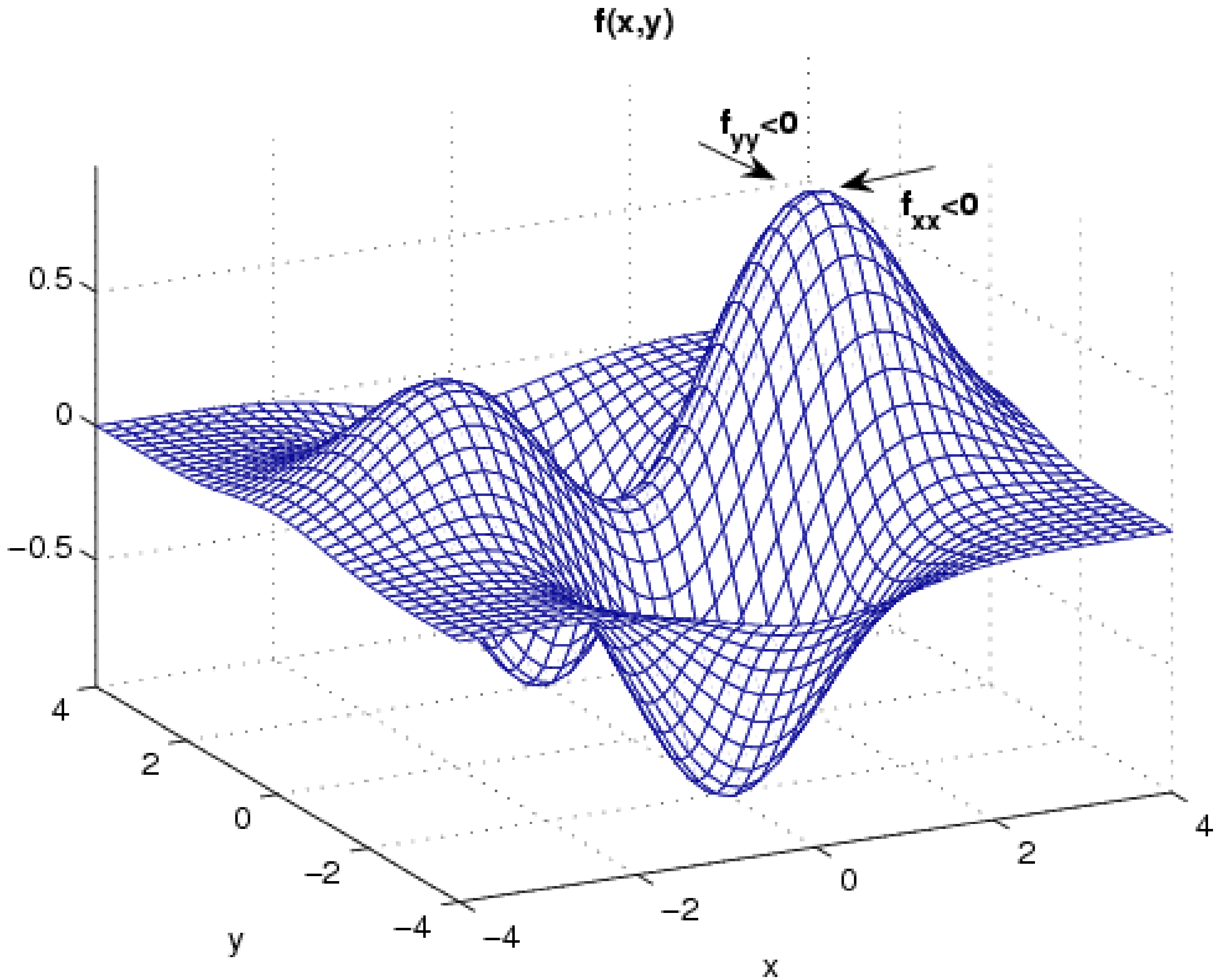
Additional sufficient second-order test

- $\frac{\partial^2 f(x,y)}{\partial x^2} < 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} < 0 \Rightarrow \text{maximum?}$
- $\frac{\partial^2 f(x,y)}{\partial x^2} < 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} < 0 \Rightarrow \text{maximum?}$
- otherwise⇒?

See the examples hereafter

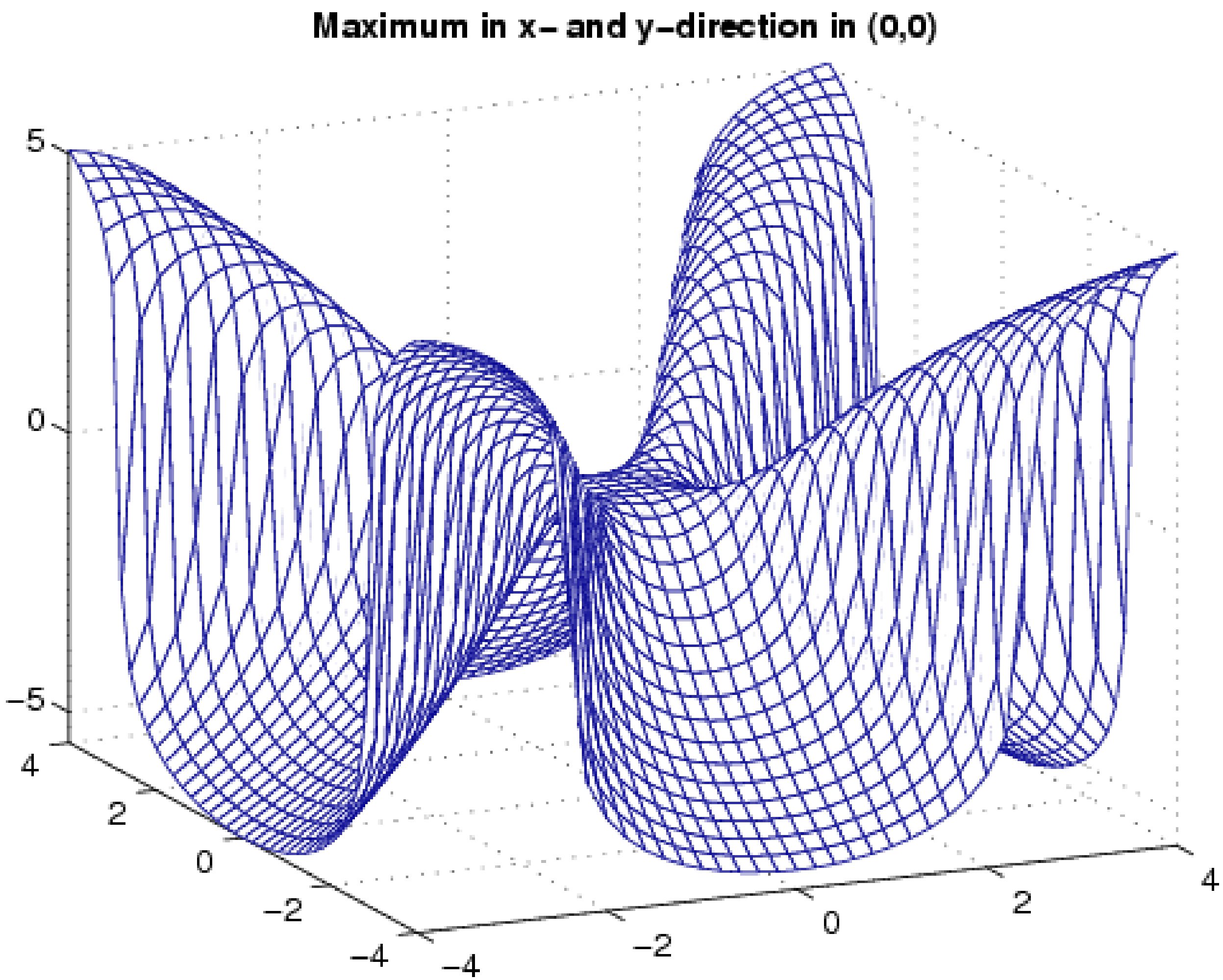
EXTREME VALUES IN TWO DIMENSIONS

In some cases: true (maximum)



EXTREME VALUES IN TWO DIMENSIONS

In other cases: false (saddle point)



MODIFIED SECOND-ORDER CONDITIONS

Additional sufficient second-order condition

- ~~not $\frac{\partial^2 f(x,y)}{\partial x^2} > 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} > 0$~~
- ~~nor $\frac{\partial^2 f(x,y)}{\partial x^2} < 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} < 0$~~

But $\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 > 0$

- more easily: $f_{xx}f_{yy} - {f_{xy}}^2 > 0$
- or: $f_{xx}f_{yy} - f_{xy}f_{yx} > 0$

These are conditions for an extreme point

- maximum point: $\frac{\partial^2 f(x,y)}{\partial x^2} < 0$ or $\frac{\partial^2 f(x,y)}{\partial y^2} < 0$
- minimum point: $\frac{\partial^2 f(x,y)}{\partial x^2} > 0$ or $\frac{\partial^2 f(x,y)}{\partial y^2} > 0$

MODIFIED SECOND-ORDER CONDITIONS

Other cases

- $\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 < 0 \Rightarrow \text{saddle point}$
- $\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 0 \Rightarrow$? (no conclusion from this method)

MODIFIED SECOND-ORDER CONDITIONS

First-order conditions for a stationary point

$$\boxed{f_x = 0 \text{ and } f_y = 0}$$

Second-order conditions:

$$\boxed{f_{xx} f_{yy} - {f_{xy}}^2 > 0 \Rightarrow \text{extreme point}}$$

- with $f_{xx} < 0 \Rightarrow \text{maximum}$ (or $f_{yy} < 0$)
- with $f_{xx} > 0 \Rightarrow \text{minimum}$ (or $f_{yy} > 0$)

$$\boxed{f_{xx} f_{yy} - {f_{xy}}^2 < 0 \Rightarrow \text{saddle point}}$$

$$\boxed{f_{xx} f_{yy} - {f_{xy}}^2 = 0 \Rightarrow ?\ (\text{no conclusion})}$$

EXAMPLE

Consider $f(x, y) = x^2 - xy + y^2 - 4y + 5$

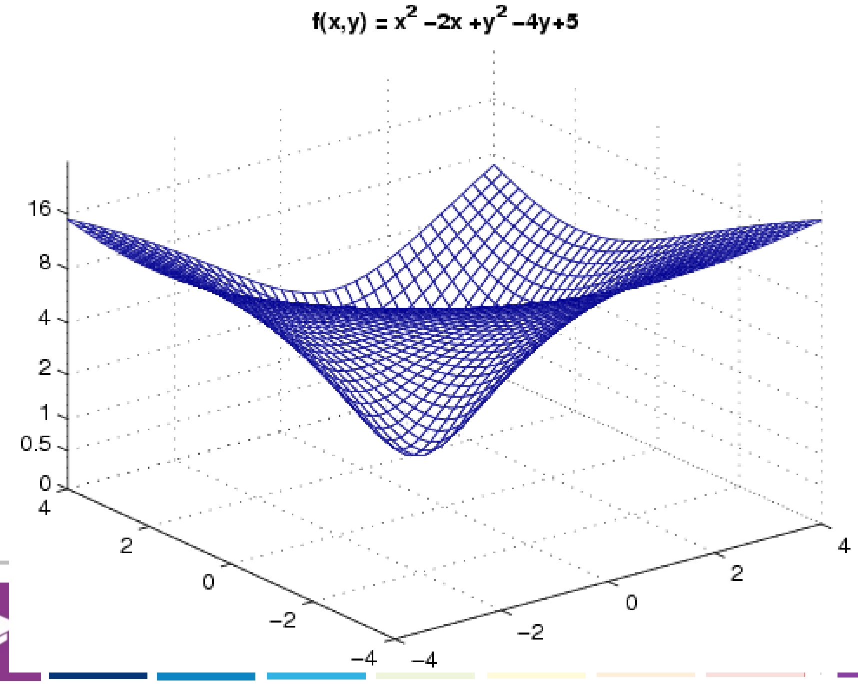
Stationary points:

- $\frac{\partial f}{\partial x} = 2x - y = 0$
- $\frac{\partial f}{\partial y} = -x + 2y - 4 = 0$
- together: $(x, y) = \left(\frac{4}{3}, \frac{8}{3}\right)$

EXAMPLE

Second-order test:

- $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2 \times 2 - (-1)^2 = 3$
- so $\left(\frac{4}{3}, \frac{8}{3}\right)$ is an extreme point
- further $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, so $\left(\frac{4}{3}, \frac{8}{3}\right)$ is a minimum point
- with minimum value $f\left(\frac{4}{3}, \frac{8}{3}\right) = -\frac{1}{3}$



EXAMPLE

Consider the function

$$f(x, y) = \frac{1}{2}x^2 e^y - \frac{1}{3}x^3 - ye^{3y}$$

Are $\left(0, -\frac{1}{3}\right)$ and $\left(e^{-\frac{1}{6}}, -\frac{1}{6}\right)$ extreme points of f?

What is the nature of the extreme points?

EXAMPLE

First, use first-order conditions to verify stationary points

- $\frac{\partial f}{\partial x} = xe^y - x^2$
- $\frac{\partial f}{\partial y} = \frac{1}{2}x^2e^y - e^{3y} - 3ye^{3y}$

Then check the nature of these points (if any) with the second-order conditions

- $\frac{\partial^2 f}{\partial x^2} = e^y - 2x$
- $\frac{\partial^2 f}{\partial y^2} = \frac{1}{2}x^2e^y - 6e^{3y} - 9ye^{3y}$
- $\frac{\partial^2 f}{\partial x \partial y} = xe^y$

OLD EXAM QUESTION

23 April 2015, Q2c

The risk of a power interruption can be decreased by several technical devices. We can install diesel aggregates with a total power capacity $D \in \mathbb{R}$ or storage batteries with a total capacity $B \in \mathbb{R}$. By doing this, the risk reduces to $R = 2D^2 - 4BD + B^4 + 2$. Find all stationary points of $R(D, B)$ and determine the nature of all those stationary points for which D and B are positive. (10 points)

OLD EXAM QUESTION

10 December 2014, Q2b

The more a guest orders, the longer he occupies a table. Excluding wine and desserts, the formula for the time (in minutes) a guest stays in the restaurant is given by

$$T = 0.2\sqrt{a} + 0.6 \ln m - 0.01(a + m)$$

where a is the expense on the appetizer and m the expense on the main course. ($a \geq 0$, $m > 0$, a and m in euro). Ignoring extreme values at the boundaries, find the extreme values of T (if any) and investigate their nature. (10 points)