Functions and equations

Business Mathematics

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CONTENTS

FUNCTIONS

A function of a variable x with domain D is a rule that assigns a unique number to each number x in D

Notation: we write the function as (for instance) f and the value assigned to (for instance) x as $f(x)$

x (the part in parantheses) is the argument of the function

Also: $x \to f(x)$

- example: $f(x) = 1 + x^2$ or $x \to 1 + x^2$ with domain $\mathbb{R}$

Domain can be restricted

- for reasons of mathematics, as in $f(x) = \frac{1}{x}$ $(x \neq 0)$
- for meaning-related reasons, as in $p(q) = 5 + q$ $(q \geq 0)$

When not specified, assume the maximum domain

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FUNCTIONS

Easily generalized to functions of two variables: $f(x, y)$
Domain is also composite, e.g. $\mathbb{R}^+ \times \mathbb{R}$, $\mathbb{R}^2$, etc.

Also: functions with a parameter: $f_a(x)$

- example: $f_a(x) = ax^2$

Also sometimes written as $f(x; a)$ or even $f(x, a)$

FUNCTIONS

Most functions are defined by a letter

- e.g., $f(x) = \frac{x-1}{x+1}$
- however, compare $x \to \frac{x-1}{x+1}$ without a letter f

In a new context, f may indicate another function

Some functions occur so often that we have a reserved symbol or notation

- e.g., $\sin x$, x^2, $x!$, $\log x$, etc.

They need not be specified, and are assumed to be known

They are often written without parantheses ($\sin x$), but with parantheses in case of doubt ($\sin(2x)$ vs. $\sin 2\, x$)

FUNCTIONS

We will assume familiarity with and actively use:

- powers (x^n) and exponentials (a^x), as well as x^y
- roots ($\sqrt[n]{x}$) and logarithms ($\log_a x$)

Further

with the special choice $n = 2$

- $\sqrt[2]{x} = \sqrt{x}$

with the special choice $a = e (= 2.7182 \cdots)$

- $e^x = \exp(x)$ and $\log_e x = \ln x$

with the special choice $a = 10$

- $\log_{10} x = \log x$

Why such a special value e?
Because $f(x) = e^x \Rightarrow f'(x) = e^x$
and because $f(x) = \ln|x| \Rightarrow f'(x) = \frac{1}{x}$

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RULES FOR SPECIAL FUNCTIONS

For “suitable” values of x, y, a, b, m, n:

- $x^{-m} = \frac{1}{x^m}$, $x^{\frac{1}{m}} = \sqrt[m]{x}$
- $x^{m+n} = x^m x^n$, $(x^m)^n = x^{mn}$
- $(\sqrt[n]{x})^m = x^{\frac{m}{n}}$, $(xy)^m = x^m y^m$
- $\log_a xy = \log_a x + \log_a y$, $\log_a x^m = m \log_a x$
- $\log_a x = \frac{\log_b x}{\log_b a}$ (in particular: $\log_a x = \frac{\ln x}{\ln b}$)
- etc.

Make sure you know them and can use them!

EQUATIONS

An <u>equation</u> is a statement of the form $A = B$, where A and B are expressions

- take for A the expression $2x + 4$ and for B the expression $18 + y$, and you get the equation $2x + 4 = 18 + y$

We use equations

- to assign variables (e.g., $p = 24$)
- to define variables (e.g., $S = pq$)
- to express relationships (e.g., $q = 20 - 3p$)

MANIPULATING EQUATIONS

Often, we need to ...

... rearrange an equation

- e.g., $2x + 4 = 18 + y \Rightarrow x = 7 + \frac{1}{2}y$

... substitute one equation into another equation

- e.g., $\begin{cases} S = pq \\ q = 20 - 3p \end{cases} \Rightarrow S = p(20 - 3p) = -3p^2 + 20p$

... solve an equation

- e.g., $4x + 3 = 23 \Leftrightarrow x = 5$

MANIPULATING EQUATIONS

Some rules for manipulating equations in scalar variables:

if $A = B$ and $B = C$ then $A = C$

- e.g., $y = 3x + 2$ and $y = -2x - 4$ then $3x + 2 = -2x - 4$

if $A = B$ then $A + C = B + C$

- e.g., $x + 5 = 12$, take $C = -5$: $x + 5 - 5 = 12 - 5$, so $x = 7$

if $A = B$ then $kA = kB$ (also valid when $k = 0$)

- e.g., $\frac{3}{x} = 2$, take $k = x$: $\frac{3}{x}x = 2x$, so $3 = 2x$
- (but check if this is permitted, i.e., if $x \neq 0$)

if $AB = 0$ then $A = 0$ or $B = 0$

- e.g., $(x - 3)(x + 4) = 0$, so $x - 3 = 0$ or $x + 4 = 0$

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MANIPULATING EQUATIONS

Important trick: write an equation as ... $\times \cdots = 0$

- example 1:

  $x^2 - 2 = 3x + 2 \Rightarrow$
  $x^2 - 3x - 4 = 0 \Rightarrow$
  $(x+1)(x-4) = 0 \Rightarrow$
  $x = -1 \vee x = 4$

- example 2:

  $2\sqrt{x} = x \Rightarrow$
  $2\sqrt{x} - x = 0 \Rightarrow$
  $\sqrt{x}(2 - \sqrt{x}) = 0 \Rightarrow$
  $\sqrt{x} = 0 \vee (2 - \sqrt{x}) = 0 \Rightarrow$
  $x = 0 \vee \sqrt{x} = 2 \Rightarrow$
  $x = 0 \vee x = 4$

Don’t use the quadratic formula when you can easily do without ...

MANIPULATING EQUATIONS

Some pitfalls:

use "=" only when left=right

- e.g., don't write $x + 3 = 5 = x = 2$ (but use $\Longrightarrow$ or $\Longleftrightarrow$)

don't lose good solutions

- e.g., $x^3 - 9x = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = 3$ (two solutions lost)

don't introduce wrong solutions

- e.g., $x = \sqrt{2x+3} \Rightarrow x^2 = 2x + 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = 3 \vee x = -1$

SOLVING EQUATIONS

Solving can mean:

calculating the value of a specified variable

- e.g., solve for x: $5x - 3 = 12$ gives $x = 3$

calculating a specified variable in terms of a parameter

- e.g., if $\frac{1}{2}x - a = 4$ with a a parameter, solving for x gives $x = 2a + 8$

calculating a set of solutions

- e.g., if $x + y = 5$, solving for y gives $y = 5 - x$

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INEQUALITIES

An <u>inequality</u> is not an equation

- even though you use an “equation editor” to type it

Take care of the rules

- e.g., if $A > B$ and $A > C$ we don’t know anything about how B compares to C
- while $A = B$ and $A = C$ implies $B = C$

Be careful about signs

- e.g., if $A < B$ we don’t know if $kA < kB$
- (because for $k = 2$ we have $3 < 4 \Rightarrow 6 < 8$, but for $k = -2$ we have $3 < 4 \Rightarrow -6 > -8$)

INEQUALITIES

Other important symbols:

- $\geq$ "greater than or equal to"
- $\leq$ "less than or equal to"
- $\neq$ "unequal to"
- $\approx$ "approximately equal to"

Mind special rules as well:

- e.g., $x \neq y \neq z \nRightarrow x \neq z$

COMMON TASKS IN EXERCISES

Solving a system of equations

- e.g., $\begin{cases} 3x + 2y = 5 \\ 4x - 3y = 12 \end{cases}$

Finding zeros of a function

- e.g., given $f(x) = 5x^2 + 4x - 9$, finding for which x is $f(x) = 0$ or $f'(x) = 0$

Finding regions where an inequality holds

- e.g., given $f(x) = 5x^2 + 4x - 9$, finding for which x is $f(x) > 0$ or $f'(x) \geq 0$

CHOICE OF SYMBOLS

It's a matter of taste

There are conventions, although they differ across disciplines

- e.g., p for price in economics, for probability in statistics

Some useful principles

- use one major symbol for "similar" quantities
- use subscripts for indicating "of what"
- e.g., p_A and p_B for price of product A and B

Greek letters an important extension

- you should know them, from α to Ω

See the document on BlackBoard

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OLD EXAM QUESTION

27 March 2015, Q1l

Find all three solutions for the equation in q: $q^3 - 2q^2 = 8q$. (exact)

OLD EXAM QUESTION

23 April 2015, Q2b

The cost C of a power black-out of duration t

For a specific supermarket, the formula is $C = 5.3(e^{12.1t} - 1)$. The manager considers leasing a diesel generator for backup power supply. The costs of this device are K (euro). Give a formula for the duration of the power interrupt t such that break-even ($C = K$) occurs. (5 points)