Gaussian elimination

Business Mathematics

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CONTENTS

THE AUGMENTED MATRIX

Consider the following set of linear equations:

$$\begin{cases} & & 2x_2 & + & 4x_3 & = & -2 \\ x_1 & - & 2x_2 & + & x_3 & = & -7 \\ 3x_1 & - & x_2 & - & 5x_3 & = & 10 \end{cases}$$

We only need to know the coefficients and right-hand term of an equation and the above systems is schematically represented as follows:

$$\left(\begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 1 & -2 & 1 & -7 \\ 3 & -1 & -5 & 10 \end{array}\right)$$

The above representation is called the augmented matrix

THE AUGMENTED MATRIX

Each row of the augmented matrix represents an equation

Therefore the following operations are admissible:

- interchanging two rows
- adding one row to another row
- multiplying a row by a number ($\neq 0$)

Using these elementary row operations we can simplify the linear system and directly find the solution(s)

ELEMENTARY ROW OPERATIONS

Start with $\left(\begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 1 & -2 & 1 & -7 \\ 3 & -1 & -5 & 10 \end{array}\right)$

- interchange rows 1 and 2: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 3 & -1 & -5 & 10 \end{array}\right)$
- subtract row 1 three times from row 3: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 0 & 5 & -8 & 31 \end{array}\right)$
- divide row 2 by 2: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \\ 0 & 5 & -8 & 31 \end{array}\right)$
- subtract row 2 five times from row 3: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -18 & 36 \end{array}\right)$
- continue ...

ELEMENTARY ROW OPERATIONS

Continue with $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -18 & 36 \end{array}\right)$

- divide row 3 by -18: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$
- subtract row 3
  two times from row 2 and one time from row 1: $\left(\begin{array}{ccc|c} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$
- add row 2 two times to row 1: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$

What does this mean?

Result: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$

- can also be written as: $\begin{cases} x_1 & & & = & 1 \\ & x_2 & & = & 3 \\ & & x_3 & = & -2 \end{cases}$

So, using elementary row operations, we have solved the system of equations

Procedure known as Gaussian elimination

GAUSSIAN ELIMINATION

General strategy for Gaussian elimination

- aim: change the original (non-augmented) matrix into an identity matrix
- way forward: make sure to obtain $0 \quad 0 \quad \cdots \quad 0 \quad 1$ in last row
- next: make sure to obtain $0 \quad 0 \quad \cdots \quad 1 \quad 0$ in penultimate row
- etc.
- finally: make sure to obtain $1 \quad 0 \quad \cdots \quad 0 \quad 0$ in first row

Basically, this is also the way computers do it

GAUSSIAN SEMI-ELIMINATION

Above we made a complete Gaussian elimination

- it is possible to do only a partial elimination

Recall the following :

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -18 & 36 \end{array}\right)$$

Stopping the elimination process at this moment also gives the possibility of quickly solving the system:

- from row 3 we can find: $-18 \times x_3 = 36 \Rightarrow x_3 = 2$
- insert the solution for x_3 in row : $x_2 - 4 = -1 \Rightarrow x_2 = 3$
- insert x_2 and x_3 in row 1: $x_1 - 6 - 2 = -7 \Rightarrow x_1 = 1$

Procedure is called <u>Gaussian semi-elimination</u>

PROBLEMS IN SOLVING A SYSTEM OF LINEAR EQUATIONS

Does Gaussian elimination always work?

Let's apply Gaussian elimination to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 3 & -6 & 3 & 10 \end{array}\right)$$

- subtract row 1 three times from row 3: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 31 \end{array}\right)$

The equation represented by row 3 has no solution

Hence, the original system has <u>no solution</u> as well

PROBLEMS IN SOLVING A SYSTEM OF LINEAR EQUATIONS

Consider another one:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 3 & -6 & 3 & -21 \end{array}\right)$$

- subtract row 1 three times from row 3: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

The equation in row 3 is now superfluous

- we can skip it, and rewrite the augmented matrix as

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -7 \\ 0 & 2 & 4 & -2 \end{array}\right)$$

PROBLEMS IN SOLVING A SYSTEM OF LINEAR EQUATIONS

Further Gaussian elimination implies

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & -9 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

- it is impossible to continue the elimination procedure

We have a free choice for one variable and choose $x_3 = \lambda$

- with λ a free parameter

Substitution in the two equations implies:

- $x_1 = -5\lambda - 9$
- $x_2 = -2\lambda - 1$

The total set of solutions for this system is:

$$\left\{ \begin{pmatrix} -5\lambda - 9 \\ -2\lambda - 1 \\ \lambda \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$$

The system has <u>infinitely many solutions</u>

THE INVERSE MATRIX

Recall that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Inverting a 3×3-matrix $\mathbf{A}$ can be thought of as solving

- $\mathbf{Ax} = \mathbf{e}_1$ (so considering $(\mathbf{A}|\mathbf{e}_1)$)
- $\mathbf{Ay} = \mathbf{e}_2$ (likewise)
- $\mathbf{Az} = \mathbf{e}_3$ (likewise)
- where $\mathbf{A}^{-1} = (\mathbf{x} \quad \mathbf{y} \quad \mathbf{z})$

This can also be written $(\mathbf{A}|\mathbf{I})$

- which then is transformed by Gaussian elimination into $(\mathbf{I}|\mathbf{A}^{-1})$

THE INVERSE MATRIX

Example

Find, if possible, the inverse matrix of the 2×2-matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$$

We start by constructing the augmented matrix

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{matrix} 4 & 1 \\ 3 & 1 \end{matrix}\middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$$

Gaussian elimination (check!) gives

$$\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 1 & -1 \\ -3 & 4 \end{matrix}\right)$$

So,

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$$

THE INVERSE MATRIX

Another example

Compute the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 6 & 12 \end{pmatrix}$$

Applying Gaussian elimination (check!) stops at

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -6 & 1 \end{array}\right)$$

It is impossible to get $\mathbf{I}$ in the front part of the augmented matrix

- so $\mathbf{A}^{-1}$ does not exist
- $\mathbf{A}$ is said to be <u>not invertible</u> ($\mathbf{A}$ is <u>singular</u>)

CASES IN SOLVING A SYSTEM OF LINEAR EQUATIONS

A system of linear equations can have

- no solution
- one unique solution
- infinitely many solutions

Which situation occurs can be found by Gaussian elimination

A square matrix $\mathbf{A}$ representing a system of linear equations $\mathbf{Ax} = \mathbf{b}$

- is invertible and has an inverse $\mathbf{A}^{-1}$ when there is one unique solution, and the solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- is not invertible (singular) in the other cases

For rectangular $\mathbf{A}$ the situation is more complicated

OLD EXAM QUESTION

22 October 2014, Q1j

Apply Gauss-Jordan elimination to the augmented system $\left(\begin{matrix}1 & 0\\3 & 5\end{matrix}\middle|\begin{matrix}3\\5\end{matrix}\right)$ and write the results in the following way: $\left(\begin{matrix}1 & 0\\0 & 1\end{matrix}\middle|\begin{matrix}\cdots\\\cdots\end{matrix}\right)$. (formula)

OLD EXAM QUESTION

10 December 2014, Q1l

It is known that all solutions of a system of linear equations are given by $\left\{\begin{pmatrix} \lambda \\ 1-2\lambda \\ 4\lambda-2 \end{pmatrix} \middle| \lambda \in \mathbb{R}\right\}$. Is the point $(-3,-5,10)$ one of the solutions? (choose one)

(A) Yes (B) No (C) There is not enough information to answer this question.