Matrices

Business Mathematics

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MATRICES

A matrix is a rectangular array of numbers or variables

Notation

- we often use bold non-italic capital letters to refer to them
- e.g., $\mathbf{Q} = \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 12.5 & -12.7 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$

Terminology

- these matrices consist of 6 respectively mn elements
- the order (or size is) 3×2 respectively $m \times n$
- when $m = n$, the matrix is a square matrix
- when $m \neq n$, the matrix is rectangular

MATRICES

Some notes on notation

- brackets: ( ) or [ ]
- indexing elements: a_{ij} or $a_{i,j}$
- representing elements: $\mathbf{A} = [A]_{ij} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ or $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
- we are quite tolerant, but please be consistent

MATRICES

In $\mathbf{Q} = \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 12.5 & -12.7 \end{pmatrix}$ the element $q_{2,1}$ refers to

Notice the order of the indices

- the cell at row 2 and column 1
- so to -2
- while $q_{1,2}$ is in row 1 and column 2 and has value 2

The order of $\mathbf{Q}$ is 3×2, not 2×3

Convention s:

- $a_{\text{row index,column index}}$
- $m_{\text{row}} \times n_{\text{column}}$
- when no ambiguity you may skip comma: a_{ij} instead of $a_{i,j}$

SPECIAL MATRICES

Zero matrix: $\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

- for a matrix of any order

Identity matrix: $\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

- for a square matrix

OPERATIONS WITH MATRICES

We can define some basic operations with matrices, similar to the basic operations with vectors

- addition ($\mathbf{A} + \mathbf{B}$, through $(a + b)_{ij} = a_{ij} + b_{ij}$)
- multiplication ($c\mathbf{A}$, through $(ca)_{ij} = c \times a_{ij}$)
- negative matrix ($-\mathbf{A}$, through $(-a)_{ij} = -a_{ij}$)
- subtraction ($\mathbf{A} - \mathbf{B}$, through $(a - b)_{ij} = a_{ij} - b_{ij}$)
- equality ($\mathbf{A} = \mathbf{B}$, through $a_{ij} = b_{ij}$)

But what about the inner product?

- not available for matrices
- instead: matrix multiplication

MATRIX MULTIPLICATION

Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices, of order $m \times p$ respectively $p \times n$

We define the matrix product $\mathbf{AB}$ as

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, i = 1, \ldots, m, j = 1, \ldots, n$$

- alternatively written as $\mathbf{A} \cdot \mathbf{B}$
- but do not write ~~$\mathbf{A} \times \mathbf{B}$~~

Notice well: the result of the multiplication of two matrices is a matrix

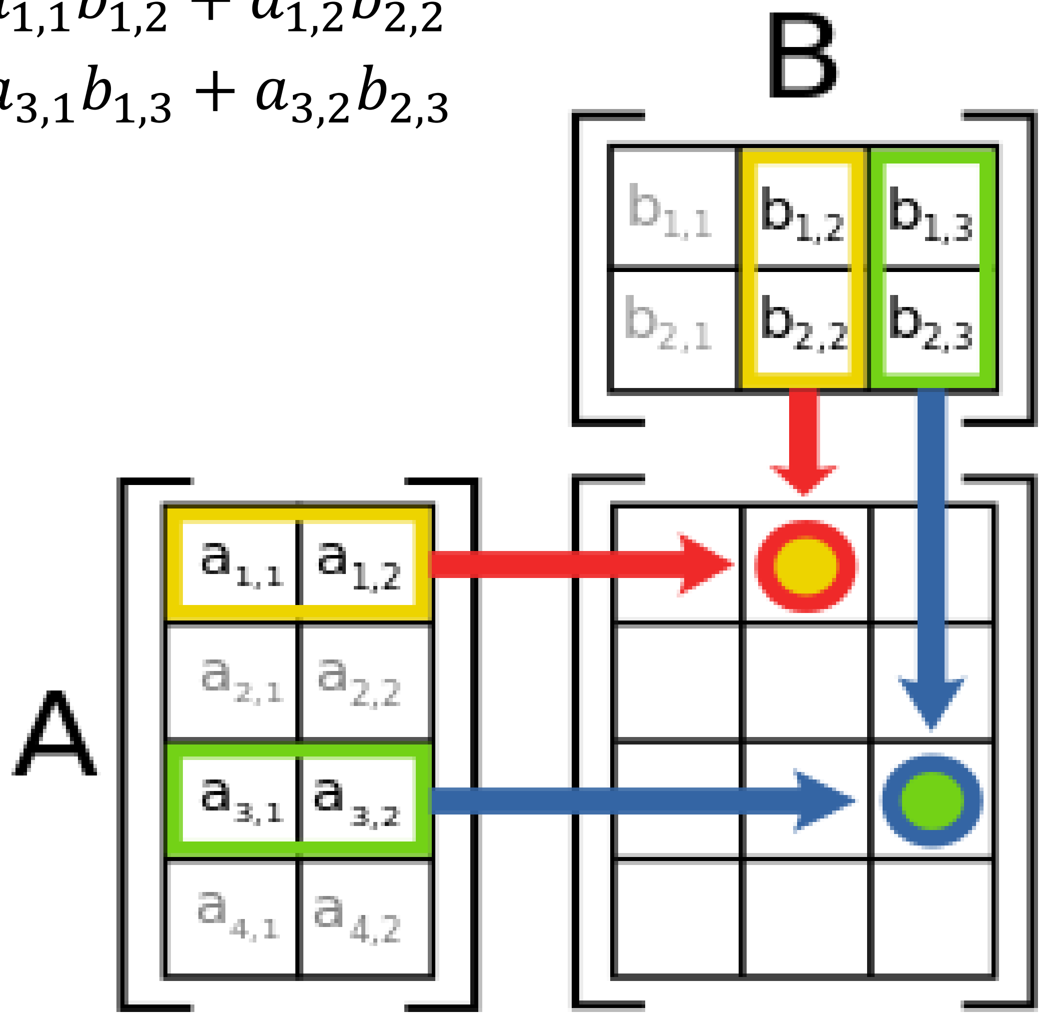
- different for the inner product of two vectors

MATRIX MULTIPLICATION

Illustration:

$(\mathbf{AB})_{1,2} = \sum_{k=1}^{2} a_{1,k} b_{k,2} = a_{1,1} b_{1,2} + a_{1,2} b_{2,2}$

$(\mathbf{AB})_{3,3} = \sum_{k=1}^{2} a_{3,k} b_{k,3} = a_{3,1} b_{1,3} + a_{3,2} b_{2,3}$



MATRIX MULTIPLICATION

Notice well the orders of the matrices:

$(\mathbf{A})_{mp}(\mathbf{B})_{pn} = (\mathbf{AB})_{mn}$

- so #columns in $\mathbf{A}$ should match #rows in $\mathbf{B}$
- and #rows in $\mathbf{AB}$ is #rows in $\mathbf{A}$
- and #columns in $\mathbf{AB}$ is #columns in $\mathbf{B}$

Consequences: given a matrix $\mathbf{A}$ or order 3×3 and a matrix $\mathbf{B}$ of order 3×2

- $\mathbf{AB}$ exists and is of order 3×2
- $\mathbf{BA}$ does not exist
- what about $\mathbf{AA}$? and $\mathbf{BB}$? and $(\mathbf{AB})(\mathbf{AB})$?

MATRIX MULTIPLICATION

It follows that (with suitable $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$)

- $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (distributive property)
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$ (associative property)

But not that

- ~~$\mathbf{AB} = \mathbf{BA}$~~ (commutative property)
- example: take $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$
- $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\mathbf{BA} = \begin{pmatrix} -22 & -44 \\ 11 & 22 \end{pmatrix}$

By the way, notice that in this example:

- $\mathbf{AB} = \mathbf{0}$, while $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$
- while for numbers $ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0$

MATRIX MULTIPLICATION

Some properties (for suitable $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$):

$\mathbf{AO} = \mathbf{0}$ and $\mathbf{OA} = \mathbf{0}$

$\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$

$\mathbf{AB} = \mathbf{AC} \not\Rightarrow \mathbf{B} = \mathbf{C}$

- example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix}$

- $\mathbf{AB} = \mathbf{AC} = \begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}$

MATRIX MULTIPLICATION

What about powers of a matrix?

Let us define $\mathbf{A}^2 = \mathbf{AA}$ for any square matrix $\mathbf{A}$

- why square?

mind the difference between "a square matrix" and "a squared matrix"

Likewise $\mathbf{A}^3 = \mathbf{AAA}$, etc.

- what is $\mathbf{AAA}$: is it $\mathbf{A}(\mathbf{AA})$ or is it $(\mathbf{AA})\mathbf{A}$?

More in general $\mathbf{A}^n = \begin{cases} \mathbf{A} & n = 1 \\ \mathbf{AA}^{n-1} & n = \{2,3,\ldots\} \end{cases}$

And what do you think of $\mathbf{A}^0$?

MORE OPERATIONS WITH MATRICES

Moral 1: every operation must be explicitly defined

Moral 2: mathematicians try to find a definition that

- is useful (why $\mathbf{AB}$ is useful will become clear later on)
- reduces to the similar operation for scalars

  Not all scalar operations have an extension to matrices

  Example:

- $\ln \mathbf{A}$
- $\frac{1}{\mathbf{A}}$
- $\sqrt{\mathbf{A}}$

We will soon introduce a sort of division by a matrix: matrix inversion

MATRIX TRANSPOSITION

Consider $\mathbf{A} = \mathbf{A}_{m \times n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$

The transpose of $\mathbf{A}$, denoted by $\mathbf{A}'$ is given by

$$\mathbf{A}' = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

In words, $\mathbf{A}'$ has n rows and m columns so $\mathbf{A}'$ is a $(n \times m)$-matrix and row i of $\mathbf{A}$ is column i of $\mathbf{A}'$

"reflection in the diagonal"

MATRIX TRANSPOSITION

Some properties (for suitable $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$):

- $(\mathbf{A}')' = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ and (therefore!) $(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$
- $(c\mathbf{A})' = c\mathbf{A}'$

SYMMETRIC MATRICES

Definition

The matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A} = \mathbf{A}'$

- so if and only if $a_{ij} = a_{ji}$ for all i, j
- note: only a square matrix can be symmetric

Example: $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}$ is symmetric

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & 5 \end{pmatrix}$ then $\mathbf{AA}'$ is symmetric

- note: $\mathbf{A}'\mathbf{A}$ is symmetric too but in general $\mathbf{A}'\mathbf{A} \neq \mathbf{AA}'$

In general $\mathbf{A}'\mathbf{A}$ is symmetric for an arbitrary matrix $\mathbf{A}$

- and so is $\mathbf{AA}'$ (why?)

You can see this for the example without even doing a calculation!

OLD EXAM QUESTION

22 October 2014, Q1d

Given is the matrix $\mathbf{R} = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 1 & 4 \\ 7 & -2 & 0 \end{pmatrix}$, and $\mathbf{A} = \mathbf{R}^2$. Calculate element $a_{3,1}$ from $\mathbf{A}$. (exact)

OLD EXAM QUESTION

10 December 2014, Q1f

A matrix $\mathbf{Q}$ of order 3×3 is known to be symmetric. Further, $q_{1,2} = 3$, $q_{1,3} = -2$, and $q_{2,3} = 0$, and $\sum_{i=1}^{3} q_{i,i} = 12$. Calculate $\sum_{i=1}^{3} \sum_{j=1}^{3} q_{i,j}$. (exact)