Multiple constrained optimization

Business Mathematics

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CONTENTS

MORE THAN TWO VARIABLES

Recall the constrained optimization problem:

$$\begin{cases} \max & f(x, y) \\ \text{s.t.} & g(x, y) = c \end{cases}$$

- function to be maximized depends on x and y

But sometimes there are more variables

- $f(x, y, z)$ or $f(x_1, x_2, x_3)$

MORE THAN TWO VARIABLES

Problem formulation:

$$\begin{cases} \max & f(x,y,z) \\ \text{s.t.} & g(x,y,z) = c \end{cases}$$

Constrained optimization problem with more than two variables

In vector notation:

$$\begin{cases} \max & f(\mathbf{x}) \\ \text{s.t.} & g(\mathbf{x}) = c \end{cases}$$

MORE THAN ONE CONSTRAINT

Again recall the constrained optimization problem:

$$\begin{cases} \max & f(x,y,z) \\ \text{s.t.} & g(x,y,z) = c \end{cases}$$

- constraint is $g(x,y,z) = c$

But sometimes there are more constraints

- $h(x,y,z) = d$ or $g_2(x,y,z) = c_2$

MORE THAN ONE CONSTRAINT

Problem formulation:

$$\begin{cases} \max & f(x,y,z) \\ \text{s.t.} & g_1(x,y,z) = c_1 \\ \text{and} & g_2(x,y,z) = c_2 \end{cases}$$

Constrained optimization problem with more than one constraint

In vector notation:

$$\begin{cases} \max & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

LAGRANGE METHOD

How to solve the problem with three variables?

$$\begin{cases} \max & f(x,y,z) \\ \text{s.t.} & g(x,y,z) = c \end{cases}$$

Generalized trick:

- introduce Lagrange multipliers λ
- define Lagrangian $\mathcal{L}(x,y,z,\lambda)$

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z) - c)$$

Find the stationary points

- by solving four equations: $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial y} = 0$, $\frac{\partial \mathcal{L}}{\partial z} = 0$, and $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$

LAGRANGE METHOD

How to solve the with two constraints?

$$\begin{cases} \max & f(x,y,z) \\ \text{s.t.} & g(x,y,z) = c \\ \text{and} & h(x,y,z) = d \end{cases}$$

Another generalized trick:

- introduce two Lagrange multipliers λ and μ
- define Lagrangian $\mathcal{L}(x,y,z,\lambda,\mu)$

$$\mathcal{L}(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda(g(x,y,z) - c) - \mu(h(x,y,z) - d)$$

Find the stationary points

- by solving five equations: $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial y} = 0$, $\frac{\partial \mathcal{L}}{\partial z} = 0$, $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$, and $\frac{\partial \mathcal{L}}{\partial \mu} = 0$

LAGRANGE METHOD

General constraint optimization problem

$$\begin{cases} \max & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

General solution with Lagrangian function

$$\boxed{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c})}$$

Stationary points of the Lagrangian function are candidate solutions to the optimization problem

- by solving many equations: $\frac{\partial \mathcal{L}(\mathbf{x},\boldsymbol{\lambda})}{\partial x_i} = 0$ and $\frac{\partial \mathcal{L}(\mathbf{x},\boldsymbol{\lambda})}{\partial \lambda_j} = 0$

THE LAGRANGE MULTIPLIER

We solve the constrained optimization problem by introducing an extra variable λ (or extra variables $\boldsymbol{\lambda}$)

Setting all partial derivatives of $\mathcal{L}$ to 0, we find the optimal value of x and y and the optimal value of f ...

... but we also find the optimal value for λ

Does it have a meaning?

THE LAGRANGE MULTIPLIER

Suppose the constant in the constraint c changes

- recall constraint equation: $g(x, y) = c$
- example, the available budget changes

Can we write the problem as a function of c?

- the optimal value of x and y depend on c $(x(c), y(c))$
- and so does the optimal value of f
- can be written as a function of c: $f\big(x(c), y(c)\big) = f^*(c)$

Theorem

$$\boxed{\frac{df^*(c)}{dc} = \lambda}$$

THE LAGRANGE MULTIPLIER

Example

Consider the production function optimization problem $f(K,L)$, with K capital and L labour:

$$\begin{cases} \max & f(K,L) = 120KL \\ \text{s.t.} & 2K + 5L = m \end{cases}$$

Introduce the Lagrangian

$$\mathcal{L}(K,L,\lambda) = 120KL - \lambda(2K + 5L - m)$$

THE LAGRANGE MULTIPLIER

Solving the first-order conditions (for given m)

$$\frac{\partial \mathcal{L}}{\partial K} = 120L - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial L} = 120K - 5\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -2K - 5L + m = 0$$

yields

$$K = K(m) = \frac{1}{4}m$$
$$L = L(m) = \frac{1}{10}m$$
$$\lambda = \lambda(m) = 6m$$

THE LAGRANGE MULTIPLIER

The maximum value at these maximum points is

$$f\big(K(m), L(m)\big) = 120KL = 120 \times \frac{1}{4}m \times \frac{1}{10}m = 3m^2$$

This has been defined above as $f^*(m)$

Now, consider $\frac{df^*(m)}{dm} = 6m$

If $m = 100$ and if m is increased by $\Delta m = 1$ to $m = 101$; then $f^*(m)$ increases approximately by

$$\underbrace{\frac{d}{dm}\bigg]_{m=100} (f^*(m))}_{=\lambda(m)} \times \Delta m = 6 \times 100 \times 1 = 600$$

- Check: $f^*(101) - f^*(100) = 3 \times 101^2 - 3 \times 100^2 = 603 \approx 600$

THE LAGRANGE MULTIPLIER

Conclusion

The value of a Lagrange multiplier gives the rate of change when the constraint constant is changed by a unit amount

OLD EXAM QUESTION

10 December 2014, Q3c

A grandmother invites her full family (children, grandchildren, etc.) for Christmas dinner. She serves wine: red (r) and white (w). The happiness she creates by serving these drinks is $H = 7R^{0.3}W^{0.7}$, where R is the number of liters of red wine, and W the number of liters of white wine. Both wines cost 5€ per liter. Because she has a small pension, her budget is limited to $5R + 5W = 250$€.

Lagrange's method yields an optimum mix of the two drinks: $R = 15$ and $W = 35$; the Lagrange multiplier is $\lambda = 0.75$, and the happiness $H = 190$ (you don't need to check these results).

Now, Grandma finds a 2 euro coin, so her budget is increased to 252€. Use a linear approximation in order to find the approximate new value of the happiness H_{new}. (8 points)

OLD EXAM QUESTION

27 March 2015, Q3d

In a more extended business case, a constrained optimization problem is introduced with 3 variables (A, B, C) and 2 constraints. The extreme values are found by introducing two Lagrange multipliers (λ and μ). The stationary points of the Lagrangian $\mathcal{L}(A, B, C, \lambda, \mu)$ are found by

$$\begin{cases} -3C - 8\lambda - 3 + 5B & = & 0 \\ 4A + 6B + 2\mu + 7 & = & 0 \\ 3\mu - 9\lambda + 1 & = & 0 \\ 4C - 2B & = & 0 \\ 2A - 7B & = & 0 \end{cases}$$

The analyst decides to solve this system of equations with matrix algebra, writing the system of equations in the form $\mathbf{Ax} = \mathbf{b}$. Specify what $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ look like. (5 points)