Partial derivatives

Business Mathematics

Faculty of Economics and Business Administration

CONTENTS

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES

Can we find the extreme values of a function $f(x,y)$ of two variable x and y?

- e.g., $f(x,y) = x^3y + x^2y^2 + x + y^2$

Differentiating with respect to x while keeping y fixed

- $\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1$

Differentiating with respect to y while keeping x fixed

- $\frac{\partial f}{\partial y} = x^3 + 2x^2y + 2y$

Clearly, in this case $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial y}$

- therefore, never write f' for a function of several variables

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES

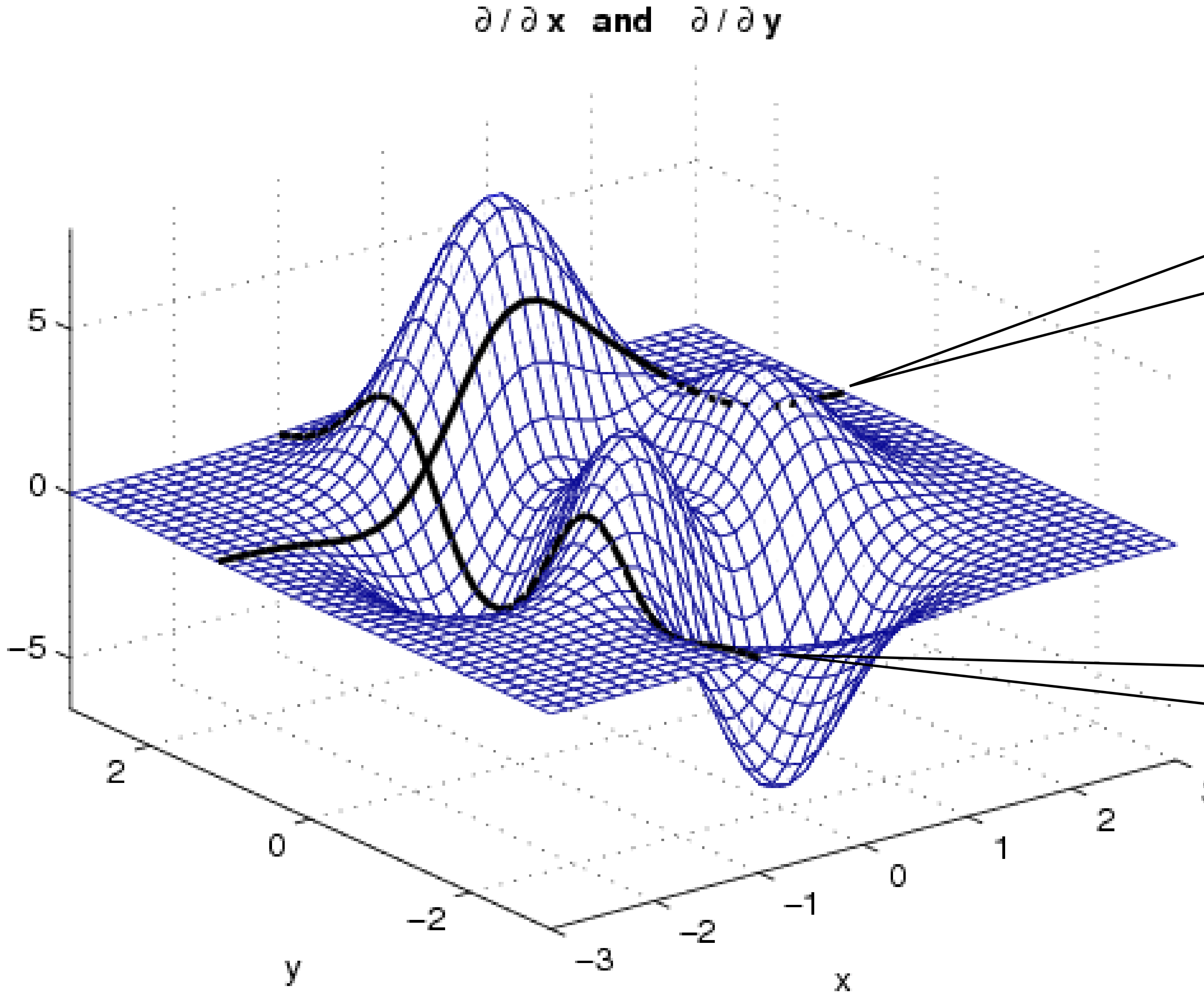
Partial derivative of f with respect to x

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

And similar with respect to y

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES



$$\frac{\partial f(x,y)}{\partial x}$$

$$\frac{\partial f(x,y)}{\partial y}$$

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES

Alternative notations

- $\frac{\partial f}{\partial x}$
- $\frac{\partial f(x,y)}{\partial x}$
- f'_x
- f'_1
- f_x
- f_1
- $\partial_x f$
- etc.

Not important to remember, but important to recognize

so, basically a lot of choice, but never write $\frac{df}{dx}$ or f'

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES

The partial derivative in a point is a number

- e.g., $\left[\frac{\partial f(x,y)}{\partial x}\right]_{(x,y)=(2,-5)} = -3$

The partial derivative over a range of points is a function of x and y

- e.g., $\frac{\partial f(x,y)}{\partial x} = 2x + 3y - 6$

DERIVATIVES FOR FUNCTIONS OF TWO VARIABLES

Example: Cobb-Douglas production function

- decribing how output depends on capital input (K) and labour input (L)

$$q(K, L) = A \times K^{\alpha} \times L^{\beta}$$

- where A, α, and β are positive constants

Marginal productivity of capital:

- $\frac{\partial q}{\partial K}$

$$\frac{\partial q}{\partial K} = A \times \alpha \times K^{\alpha - 1} \times L^{\beta}$$

- when $0 < \alpha < 1$, $\frac{\partial q}{\partial K}$ is decreasing
- diminishing marginal returns

HIGHER-ORDER PARTIAL DERIVATIVES

Second-order: four possibilities:

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

Alternative notations:

- $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial f(x,y)}{\partial x \partial y}, f''_{yx}, f''_{21}, f_{yx}, f_{21}, \partial_{xy} f$, etc.

so, never $\frac{d^2 f}{dx^2}$ or f''

HIGHER-ORDER PARTIAL DERIVATIVES

Example: $f(x,y) = x^3y + x^2y^2 + x + y^2$

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 6xy + 2y^2$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = 2x^2 + 2$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4xy$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4xy$

For almost all functions $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$

- and certainly for all functions we encounter in business and economics

HIGHER-ORDER PARTIAL DERIVATIVES

Likewise, we can define third-order

- $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial x^3}$
- $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$
- how many are there?
- how many are different?

And even higher-order partial derivatives

- $\frac{\partial^n f}{\partial x^n}$
- $\frac{\partial^n f}{\partial x^{n-1} \partial y}$
- etc.

DERIVATIVES FOR FUNCTIONS OF MANY VARIABLES

Let $f(x_1, x_2, x_3, \ldots, x_n)$

We can form n first-order partial derivatives

- $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$

and many many second-order partial derivatives

ELASTICITY FOR FUNCTIONS OF TWO VARIABLES

Recall the definition of elasticity for $f(x)$

- $\mathrm{El}_x f(x) = \frac{x}{f(x)} \frac{df(x)}{dx}$

Easily generalized to functions of two variables:

- $\mathrm{El}_x f(x,y) = \frac{x}{f(x,y)} \frac{\partial f(x,y)}{\partial x}$
- $\mathrm{El}_y f(x,y) = \frac{y}{f(x,y)} \frac{\partial f(x,y)}{\partial y}$

now, you understand why the subscript at El_* is there

Example for Cobb-Douglas

- $q(K,L) = A \times K^{\alpha} \times L^{\beta}$
- $\mathrm{El}_K q(K,L) = \frac{K}{q(K,L)} \frac{\partial q(K,L)}{\partial K} = \frac{K^{1-\alpha} \times L^{-\beta}}{A} A \times \alpha K^{\alpha-1} \times L^{\beta} = \alpha$
- independently of K and L, constant elasticity of substitution

OLD EXAM QUESTION

27 March 2015, Q1b

Given is the function $f(p,q) = e^{p^2+pq}$. Calculate $\frac{\partial f}{\partial p}$ at $(p,q) = (1,2)$. (exact)

OLD EXAM QUESTION

22 October 2014, Q1h

Function q is given by $q(a,b) = a^2b - 3ab^2$. Determine $\frac{\partial^2 q}{\partial a \partial b}$. (formula)