Systems of linear equations

Business Mathematics

VU University Amsterdam
Faculty of Economics and Business Administration

CONTENTS

Systems of linear equations

Matrix formulation

Matrix inversion

Solving a system of linear equations

Old exam question

SYSTEMS OF LINEAR EQUATIONS

One linear equation in one unknown

- $5x = 12$
- can be solved as $x = \frac{12}{5}$

Two simultaneous linear equations in two unknowns

- $\begin{cases} 4x + 3y & = & 20 \\ -2x + 4y & = & 12 \end{cases}$
- can be solved as $\begin{cases} x & = & 2 \\ y & = & 4 \end{cases}$

And so on ...

More in general ...

SYSTEMS OF LINEAR EQUATIONS

n simultaneous linear equations in n unknowns

- $$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n & = & b_2 \\ \cdots & = & \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n & = & b_n \end{cases}$$
- can (sometimes?) be solved as $\begin{cases} x_1 & = & ? \\ x_2 & = & ? \\ \cdots & = & \cdots \\ x_n & = & ? \end{cases}$

But how? What are the question marks? And when?

SYSTEMS OF LINEAR EQUATIONS

We define a <u>system of linear equations</u> of order n by

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n & = & b_2 \\ \cdots & = & \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n & = & b_n \end{cases}$$

It has

- n equations ($=$-signs)
- n unknowns (x-values)
- $n^2 + n$ known coefficients (a-values and b-values)

It can also be written as

$$\sum_{j=1}^{n} a_{i,j}x_j = b_i, i = 1, \ldots, n$$

MATRIX FORMULATION

The system of linear equations

$$\sum_{j=1}^{n} a_{i,j} x_j = b_i, i = 1, \dots, n$$

can concisely be written in matrix form as

$$\boxed{\mathbf{Ax} = \mathbf{b}}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}; \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix};$$

MATRIX FORMULATION

Previous example revisited

- $\begin{cases} 4x + 3y & = & 20 \\ -2x + 4y & = & 12 \end{cases}$

In matrix terms:

- $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 20 \\ 12 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$
- so indeed $\mathbf{Ax} = \mathbf{b}$

MATRIX FORMULATION

Known (=given):

- the $n \times n$-matrix $\mathbf{A}$
- the n-vector $\mathbf{b}$

Unknown (=to be solved):

- the n-vector $\mathbf{x}$

Main problem:

- how to find the solution $\mathbf{x}$?
- how to change the equation $\mathbf{Ax} = \mathbf{b}$ into a solution $\mathbf{x} = \cdots$?

MATRIX INVERSION

Start with the linear equation $ax = b$ with $a(\neq 0)$ and b numbers

How to change into a solution of the form $x = \cdots$?

- premultiply both sides by $a^{-1} = \frac{1}{a}$
- this yields $a^{-1}(ax) = a^{-1}b$
- by virtue of the associative law, this is $(a^{-1}a)x = a^{-1}b$
- and because $a^{-1}a = 1$, we have $x = a^{-1}b$

So if we know what a^{-1} is, we have a general recipe for solving x

Can we generalize this?

MATRIX INVERSION

A matrix $\mathbf{B}$ is called the inverse of the $n \times n$ matrix $\mathbf{A}$ if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$

- we denote $\mathbf{B}$ by $\mathbf{A}^{-1}$

So, when $\mathbf{A}$ is an invertible $n \times n$-matrix, with inverse matrix $\mathbf{A}^{-1}$, we have

$$\boxed{\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n}$$

MATRIX INVERSION

Some properties, assuming that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ and invertible:

- $\mathbf{A}^{-1}$ is of order $n \times n$
- $\mathbf{A}^{-1}$ is unique
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}\ (c \neq 0)$

SOLVING A SYSTEM OF LINEAR EQUATIONS

Suppose $\boxed{\mathbf{Ax} = \mathbf{b}}$ is a system of linear equations with $\mathbf{x}$ unknown, and with $\mathbf{A}$ invertible

Then premultiply by $\mathbf{A}^{-1}$:

- $\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}$

Apply the associative property for matrices:

- $(\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Apply the defining property of the inverse matrix:

- $\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$

Apply a property of the identity matrix:

- $\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$

So, when $\mathbf{A}^{-1}$ is known, we have solved $\mathbf{x}$!

SOLVING MATRIX EQUATIONS

Solve $\mathbf{X}$ from $\mathbf{AX} - \mathbf{B} = \mathbf{C}^2\mathbf{X}$

- with $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ square matrices

Solution

- $\mathbf{AX} - \mathbf{B} = \mathbf{C}^2\mathbf{X} \Rightarrow$
- $\mathbf{AX} - \mathbf{C}^2\mathbf{X} = \mathbf{B} \Rightarrow$
- $(\mathbf{A} - \mathbf{C}^2)\mathbf{X} = \mathbf{B} \Rightarrow$
- $\mathbf{X} = (\mathbf{A} - \mathbf{C}^2)^{-1}\mathbf{B}$ (provided $(\mathbf{A} - \mathbf{C}^2)$ is invertible)

Example (check the details!)

- $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; $\mathbf{C} = \mathbf{I}_2$
- $(\mathbf{A} - \mathbf{C}^2)^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$; $\mathbf{X} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{6} \end{pmatrix}$

FINDING THE INVERSE MATRIX

We have defined the inverse matrix through a property ($\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$)

But we still do not know how to calculate it

- Compare the definition of the square root: $\sqrt{a}$ is the non-negative number such that $(\sqrt{a})^2 = a$

Different ways:

- by hand (Gaussian elimination, later on)
- Excel (work groups, later on)
- dedicated software (Matlab, R, etc., not in this course)

OLD EXAM QUESTION

22 October 2014, Q2a

A macro-economic model is defined by the equations

$$\begin{aligned} Y &= C + S + I \\ C &= C_0 + cY \\ S &= S_0 + sY \\ I &= iY \end{aligned}$$

where Y, C, S, and I are variables, and C_0, S_0, c, s, and i are constants. Write the model in matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{x} = \begin{pmatrix} Y \\ C \\ S \\ I \end{pmatrix}$ and $\mathbf{A}$ and $\mathbf{b}$ containing constants only. (5 points)

OLD EXAM QUESTION

10 December 2014, Q1b

Given is the matrix equation $\mathbf{YX}' = \mathbf{YQY}$, where $\mathbf{Y}$ and $\mathbf{Q}$ are invertible square matrices. Give an expression for $\mathbf{X}$ that is simplified as much as possible. (formula)