Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Tutorial 4 – Exercises

Instruction

In a tutorial session of 2 hours, we will obviously not be able to discuss all questions. Therefore, the following procedure applies:

- we expect students to prepare all exercises in advance;
- we will dicuss only a selection of exercises;
- exercises that were not discussed during class are nevertheless part of the course;
- students can indicate their wish list of exercises to be discussed during the session;
- teachers may invite students to answer questions, orally or on the blackboard.

!!! We further understand that your time is limited, and in particular that your time between lecture and tutorial may be limited. In case you have no time to prepare everything, we kindly advise you to give priority to the exercises that are indicated with the !!! sign.

Extreme values in two dimensions

!!! Q1 (Sydsæter & Hammond, 4/E, 13.1.1)
The function f defined for all (x, y) by $f(x, y) = -2x^2 - y^2 + 4x + 4y - 3$ has a maximum. Find the corresponding values of x and y.

!!! Q2 (based on Sydsæter & Hammond, 4/E, 13.1.3)
In a profit-maximizing problem, let the production function be given as
$Q = F(K, L) = 80 - (K - 3)^2 - 2(L - 6)^2 - (K - 3)(L - 6)$,
where Q is output, K is capital input, and L is labour input. The price per unit of output is $p = 1$, the cost (or rental) per unit of capital is $r = 0.65$, and the wage rate is $w = 1.2$. Find the only possible values of K and L that maximize profits.

Q3 (based on Sydsæter & Hammond, 4/E, 13.2.3)
Solve the utility-maximizing problem $\max U = xyz$ subject to $x + 3y + 4z = 108$, by making U a function of y and z by eliminating the variable x. Assume: $x > 0$, $y > 0$, and $z > 0$.

Q4 (Sydsæter & Hammond, 4/E, 13.2.5)
A firm produces two goods. The cost of producing x units of good 1 and y units of good 2 is
$C(x, y) = x^2 + xy + y^2 + x + y + 14$
Suppose the firm sells all its outputs of each good at prices per unit of p an q respectively. Find the values of x and y that maximize profits. (Assume $\frac{1}{2}p + \frac{1}{2} < q < 2p - 1$ and $p > 1$.)

Q5 (Sydsæter & Hammond, 4/E, 13.3.1)
(a) Find the partial derivatives of the first and second order for the function f defined for all (x, y) by $f(x, y) = 5 - x^2 + 6x - 2y^2 + 8y$.
(b) Find the only stationary point and classify it by using the second-order derivative test.

Q6 Compute the partial derivatives of the first and second order of $f(x, y) = x^3 + 2xy - 5x - y^2$ and find the stationary points and classify them.

!!! Q7 Consider the function f defined for all (x, y) by $f(x, y) = \frac{1}{2}x^2 - x + ay(x - 1) - \frac{1}{3}y^3 + a^2y^2$, where a is a constant. Find all the stationary points of f and classify them.

Systems of linear equations

!!! Q1 (Sydsæter & Hammond, 4/E, 16.6.1.a)

Prove that $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}$

Q2 (Sydsæter & Hammond, 4/E, 16.6.1.b)

Prove that $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 8/7 & -1 & 3/7 \\ -2/7 & 0 & 1/7 \end{pmatrix}$

Q3 (Sydsæter & Hammond, 4/E, 16.6.4)

Let $\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$. Show that $\mathbf{A}^3 = \mathbf{I}_2$. Use this to find $\mathbf{A}^{-1}$.

Q4 The two 2×2-matrices $\mathbf{C}$ and $\mathbf{D}$ are given by: $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$. Further, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let X be an unknown 2×2-matrix. Solve $\mathbf{X}$ from the following matrix equation: $\mathbf{XC} = \mathbf{C} + \mathbf{D}^{-1}$.

!!! Q5 Let $\mathbf{B}$ be a 2×2-matrix: $\mathbf{B} = \begin{pmatrix} 0 & \frac{1}{2}b \\ 1 & 0 \end{pmatrix}$. For what value(s) of b does the matrix equation $\mathbf{B}^{-1} = \mathbf{B}$ hold?

Q6 (based on Sydsæter & Hammond, 4/E, 15.6.1.b)

Verify that $\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

Next use this result to solve the following system $\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_1 - x_2 + x_3 = 5 \\ 2x_1 + 3x_2 - x_3 = 1 \end{cases}$.

Curve fitting

Q1 Given are five data points $(3,8)$, $(4,10)$, $(5,11)$, $(6,14)$, and $(7,15)$. Determine the optimal line that fits these point, using the least squares criterion. Tip: you can use Excel to speed up the calculations, but do not use the built-in trend-line, covariance, etc., but program the expression with sums.

!!! Q2 Given is the regresssion line $y = -15 + 210x$, where x is measured in km^2, and y is measured in euro.

(a) What happens with the regression coefficients -15 and 210 when we decide to measure y in keuro instead of in euro?

(b) And what happens if we measure x in m^2 instead of km^2?

(c) Develop a general formula for what happens with regession coefficient a and b in the regression line $y = a + bx$, when x is changed into αx and y is changed into βy.

Q3 A data set with $n = 100$ gives a regresssion line $y = 15 + 210x$. Which statements are true?

(A) There are points with a negative y-value.

(B) The covariance of x and y is positive.
(C) The mean of y is bigger than the mean of x.
(D) Almost all data points are in the first or third qudrant.
(E) None of the above statements is true.

Q4 Does the OLS procedure always produce a best line estimate? Interpret any exception.

Q5 Consider again the five data points $(3,8)$, $(4,10)$, $(5,11)$, $(6,14)$, and $(7,15)$. Write the regression problem in matrix form.

Gaussian elimination

!!! Q1 (Sydsæter & Hammond, 4/E, 15.6.1.b)

Solve the following system by Gaussian elimination: $\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_1 - x_2 + x_3 = 5 \\ 2x_1 + 3x_2 - x_3 = 1 \end{cases}$.

!!! Q2 (Sydsæter & Hammond, 4/E, 15.6.1.c)

Solve the following system by Gaussian elimination: $\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$.

Q3 (Sydsæter & Hammond, 4/E, 15.6.2)

Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b: $\begin{cases} x + y - z = 1 \\ x - y + 2z = 2 \\ x + 2y + az = b \end{cases}$

Q4 (Sydsæter & Hammond, 4/E, 15.Review.9)

Use the Gaussian elimination to find for what values of a the following system has solutions. Then find all the possible solutions.

$$\begin{cases} x + ay + 2z = 0 \\ -2x - ay + z = 4 \\ 2ax + 3a^2y + 9z = 4 \end{cases}$$

!!! Q5 (based on Sydsæter & Hammond, 4/E, 16.7.5.a)

Use Gauss-Jordan elimination to calculate the inverse (provided it exists) of the following matrix: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

!!! Q6 Let $\mathbf{A}$ be a 2×2-matrix: $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 4 & a \end{pmatrix}$. For what value(s) of a does the inverse matrix $\mathbf{A}^{-1}$ not exist?