Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Tutorial 3 – Full solutions

Instruction

In a tutorial session of 2 hours, we will obviously not be able to discuss all questions. Therefore, the following procedure applies:

- we expect students to prepare all exercises in advance;
- we will dicuss only a selection of exercises;
- exercises that were not discussed during class are nevertheless part of the course;
- students can indicate their wish list of exercises to be discussed during the session;
- teachers may invite students to answer questions, orally or on the blackboard.

!!! We further understand that your time is limited, and in particular that your time between lecture and tutorial may be limited. In case you have no time to prepare everything, we kindly advise you to give priority to the exercises that are indicated with the !!! sign.

Partial derivatives

!!! Q1 (Sydsæter & Hammond, 4/E, 11.2.1.c)
Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following: $z = x^3y^4$

A1 $\frac{\partial z}{\partial x} = 3x^2y^4$ and $\frac{\partial z}{\partial x} = 4x^3y^3$

Q2 (Sydsæter & Hammond, 4/E, 11.2.2.e)
Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following: $z = e^{xy}$

A2 $\frac{\partial z}{\partial x} = ye^{xy}$ and $\frac{\partial z}{\partial y} = xe^{xy}$

Q3 (Sydsæter & Hammond, 4/E, 11.2.2.h)
Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following: $z = \ln xy$

A3 $\frac{\partial z}{\partial x} = \frac{1}{x}$ and $\frac{\partial z}{\partial y} = \frac{1}{y}$

Sol $\frac{\partial z}{\partial x} = \frac{1}{xy}y = \frac{1}{x}$ and $\frac{\partial z}{\partial y} = \frac{1}{xy}x = \frac{1}{y}$

!!! Q4 (Sydsæter & Hammond, 4/E, 11.2.4.b)
Find all first- and second-order partial derivatives for the following: $z = x^3y^2$

A4 $\frac{\partial z}{\partial x} = 3x^2y^2, \frac{\partial z}{\partial y} = 2x^3y, \frac{\partial^2 z}{\partial x^2} = 6xy^2, \frac{\partial^2 z}{\partial y^2} = 2x^3, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 6x^2y$

Q5 (Sydsæter & Hammond, 4/E, 11.2.9)
If a household consumes x units of one good and y units of a second good, its satisfaction is measured by the function $s(x, y) = 2\ln x + 4\ln y$. Suppose that the household presently consumes 20 units of the first good and 30 units of the second.

(a) What is the approximate increase in satisfaction from consuming one extra unit of the first good?

(b) What is the approximate increase in satisfaction from consuming one extra unit of the second good?

A5 (a) $\frac{1}{10}$ (b) $\frac{2}{15}$

Sol (a) $s(x, y) = 2\ln x + 4\ln y \Rightarrow \frac{\partial s}{\partial x} = \frac{2}{x} = \frac{2}{20} = \frac{1}{10}$ (b) $s(x, y) = 2\ln x + 4\ln y \Rightarrow \frac{\partial s}{\partial y} = \frac{4}{y} = \frac{4}{30} = \frac{2}{15}$

Indefinite integrals

!!! Q1 (Sydsæter & Hammond, 4/E, 6.6.6.a)
For the following function, find a function $F(x)$ that has $f(x)$ as its derivative, that is $F'(x) = f(x)$. $f(x) = x^2$
(Note that you are not asked to find $f'(x)$.)

A1 $\frac{1}{3}x^3 + C$

Sol $f(x) = x^2 \Rightarrow F(x) = \int x^2 dx = \frac{1}{3}x^3 + C$

Extra Stress the importance of the dx in the integral.

Q2 (Sydsæter & Hammond, 4/E, 9.1.1.b)
Find the following integral by using $\int x^a \, dx = \frac{1}{a+1}x^{a+1} + C \quad (a \neq 1)$: $\int x\sqrt{x}dx$

A2 $\frac{2}{5}x^2\sqrt{x} + C$

Sol $F(x) = \int x\sqrt{x}dx = \int x^{3/2}dx = \frac{2}{5}x^{5/2} + C = \frac{2}{5}x^2\sqrt{x} + C$

!!! Q3 (Sydsæter & Hammond, 4/E, 9.1.6.a)
Show that $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

Sol Show this by differentiating the right side of this expression:
$\frac{d}{dx}\left(\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C\right) = x^2 \ln x + \frac{1}{3}x^3 \cdot \frac{1}{x} - \frac{1}{3}x^2 + 0 = x^2 \ln x + \frac{1}{3}x^2 - \frac{1}{3}x^2 = x^2 \ln x$

Extra This is an obvious case where differentiating is much easier than integrating.

!!! Q4 Suppose we know that the marginal costs of a company for producing an output q are given by $MC(q) = 3q^{\frac{4}{5}}$, and that the start-up costs for producing nothing are 20. Find the total costs TC.

A4 $TC(q) = \frac{5}{3}q^{\frac{9}{5}} + 20$

Sol $TC(q) = \int MC(q)dq = \int 3q^{\frac{4}{5}}dq = \frac{5}{3}q^{\frac{9}{5}} + C$. It is given that $TC(0) = 20$, hence $C = 20$. So $TC(q) = \frac{5}{3}q^{\frac{9}{5}} + 20$

Definite integrals

Q1 (Sydsæter & Hammond, 4/E, 9.2.8.d)
Evaluate the integral: $\int_{-2}^{-1} \frac{1}{y} dy$

A1 $-\ln 2$

Sol $\int_{-2}^{-1} \frac{1}{y} dy = [\ln|y|]_{-2}^{-1} = \ln|-1| - \ln|-2| = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$

Extra Draw attention to the fact that you should check if the function is defined over the full interval. If the integration boundaries were $[-2, +1]$, the answer is wrong.

!!! Q2 (Sydsæter & Hammond, 4/E, 9.2.8.b)
Evaluate the integral: $\int_{-3}^{-1} \xi^2 d\xi$

A2 $\frac{26}{3}$

Sol $\int_{-3}^{-1} \xi^2 d\xi = \left[\frac{1}{3}\xi^3\right]_{-3}^{-1} = -\frac{1}{3} - (-9) = \frac{26}{3}$

Extra Many students find this one much more difficult than $\int_{-3}^{-1} x^2 dx$, because they're afraid of the weird symbol. Please stress that we have a link to the Greek alphabet on our Topics list, and that they should practice it, because we often see unreadable scribbles at the exam.

!!! Q3 (Sydsæter & Hammond, 4/E, 9.3.4.a)

Evaluate the integral $\int_0^1 x^p(x^q + x^r)dx$, where p, q, and r are positive constants.

A3 $\frac{1}{p+q+1} + \frac{1}{p+r+1}$

Sol $\int_0^1 x^p(x^q + x^r)dx = \int_0^1 (x^p x^q + x^p x^r)dx = \int_0^1 (x^{p+q} + x^{p+r})dx = \left[\frac{x^{p+q+1}}{p+q+1} + \frac{x^{p+r+1}}{p+r+1}\right]_0^1 =$
$\frac{1}{p+q+1} + \frac{1}{p+r+1}$

!!! Q4 (Sydsæter & Hammond, 4/E, 9.3.4.b)

Find the function $f(x)$ if $f'(x) = ax^2 + bx$, and (i) $f'(1) = 6$ (ii) $f''(1) = 18$ (iii) $\int_0^2 f(x)dx = 18$

A4 $f(x) = 4x^3 - 3x^2 + 5$

Sol $f'(x) = ax^2 + bx$

$f'(1) = a + b = 6 \Rightarrow b = 6 - a$

$f''(x) = 2ax + b \Rightarrow f''(1) = 2a + b = 18 \Rightarrow b = 18 - 2a$

So: $6 - a = 18 - 2a \Rightarrow a = 12 \Rightarrow b = 6 - 12 = -6 \Rightarrow f'(x) = 12x^2 - 6x \Rightarrow f(x) = 4x^3 - 3x^2 + C$

$\int_0^2 (4x^3 - 3x^2 + C)dx = [x^4 - x^3 + Cx]_0^2 = 8 + 2C - 0 = 8 + 2C = 18 \Rightarrow C = 5$

$f(x) = 4x^3 - 3x^2 + 5$

Extra This exercise is a good training in mathematical strategy: what do we want to achieve, and which steps must we take and remember.

!!! Q5 (Sydsæter & Hammond, 4/E, 9.3.7.a)

Find: $\frac{d}{dt}\int_0^t x^2 dx$

A5 t^2

Sol $\frac{d}{dt}\int_0^t x^2 dx = \frac{d}{dt}\left(\left[\frac{1}{3}x^3\right]_0^t\right) = \frac{d}{dt}\left(\frac{1}{3}t^3\right) = t^2$ (this is the hard way!)

Short: the antiderivative of $f(x) = x^3$ is $F(x)$ then $\frac{d}{dt}\int_0^t x^2 dx = \frac{d}{dt}\{F(t) - F(0)\} = f(t) = t^2$

Extra You may stress the elegance of this second approach: without calculating F, you still use it as if you knew it. That's part of the power of mathematics!

Q6 (Sydsæter & Hammond, 4/E, 9.3.7.b)

Find: $\frac{d}{dt}\int_t^3 e^{-x^2} dx$

A6 $-e^{-t^2}$

Sol Now the hard way is impossible. Use the short way (Always do!)

$\frac{d}{dt}\int_t^3 e^{-x^2} dx$; one cannot find an antiderivative $F(x)$ of $f(x) = e^{-x^2}$; use $F(x)$ although unknown:

$\frac{d}{dt}\int_t^3 e^{-x^2} dx = \frac{d}{dt}\{F(3) - F(t)\} = 0 - f(t) = -e^{-t^2}$

Matrices

Q1 (based on Sydsæter & Hammond, 4/E, 15.1.1.d)

Decide if the following single equation is linear or not: $3.33x - 4y + \frac{800}{3}z = 3$

A1 Yes

Q2 (based on Sydsæter & Hammond, 4/E, 15.1.1.b)

Decide if the following single equation is linear or not: $\sqrt{3}x + 8xy - z - w = 0$

A2 No

Sol Because xy, the answer is No

!!! Q3 (Sydsæter & Hammond, 4/E, 15.1.6)

T.Haavelmo devised a model of the US economy for the years 1929-1941 based on the following equations:

(i) $c = 0.712y + 95.05$

(ii) $s = 0.158(c + x) - 34.30$

(iii) $y = c + x - s$

(iv) $x = 93.53$

Here x denotes total investment, y is disposable income, s is the total saving by firms, and c is total consumption. Write the system of equations as a linear system of equations when the variables appear in the order x, y, s, and c.

A3
$$\begin{cases} & - & 0.712y & & & + & c & = & 95.05 \\ 0.158x & & & - & s & + & 0.158c & = & 34.30 \\ x & - & y & - & s & + & c & = & 0 \\ x & & & & & & & = & 95.53 \end{cases}$$

Sol
$$\begin{cases} c = 0.712y + 95.05 \\ s = 0.158(c + x) - 34.30 \\ y = c + x - s \\ x = 93.53 \end{cases} \Rightarrow \begin{cases} & - & 0.712y & & & + & c & = & 95.05 \\ 0.158x & & & - & s & + & 0.158c & = & 34.30 \\ x & - & y & - & s & + & c & = & 0 \\ x & & & & & & & = & 95.53 \end{cases}$$

Do not do the solving of this system.

!!! Q4 For what values of u and v does $\begin{pmatrix} (1-u)^2 & v^2 & 3 \\ v & 2u & 5 \\ 6 & u & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & u \\ v & -3v & u-v \\ 6 & v+5 & -1 \end{pmatrix}$?

A4 $u = 3$ and $v = -2$

Sol The position (1,3) implies: $u = 3$. Then, the position (2,3) implies $u - v = 5$, so $v = -2$. The other elements then need to be checked.

Q5 (Sydsæter & Hammond, 4/E, 15.2.2)

Evaluate $\mathbf{A} + \mathbf{B}$ and $3\mathbf{A}$ when $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$.

A5 $\begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$ and $\begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$

Sol $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 0+1 & 1-1 \\ 2+5 & 3+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$

$3\mathbf{A} = \mathbf{3}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 0 & 3 \cdot 1 \\ 3 \cdot 2 & 3 \cdot 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$

!!! Q6 (Sydsæter & Hammond, 4/E, 15.3.1.b)

Compute the products **AB** and **BA**, if possible, for the following:

$\mathbf{A} = \begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix}$.

A6 $\mathbf{AB} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$; $\mathbf{BA} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

Sol $\mathbf{AB} = \begin{pmatrix} 8 \cdot 2 + 3 \cdot 4 + -2 \cdot 1 & 8 \cdot -2 + 3 \cdot 3 + -2 \cdot -5 \\ 1 \cdot 2 + 0 \cdot 4 + 4 \cdot 1 & 1 \cdot -2 + 0 \cdot 3 + 4 \cdot -5 \end{pmatrix} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$

$\mathbf{BA} = \begin{pmatrix} 2 \cdot 8 + -2 \cdot 1 & 2 \cdot 3 + -2 \cdot 0 & 2 \cdot -2 + -2 \cdot 4 \\ 4 \cdot 8 + 3 \cdot 1 & 4 \cdot 3 + 3 \cdot 0 & 4 \cdot -2 + 3 \cdot 4 \\ 1 \cdot 8 + -5 \cdot 1 & 1 \cdot 3 + -5 \cdot 0 & 1 \cdot -2 + -5 \cdot 4 \end{pmatrix} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

Q7 (Sydsæter & Hammond, 4/E, 15.3.1.c)
Compute the products **AB** and **BA**, if possible, for the following:

$\mathbf{A} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ and $\mathbf{B} = (0, -2, 3)$.

A7 $\mathbf{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -8 & 12 \end{pmatrix}$; $\mathbf{BA} = 16$

Sol $\mathbf{AB} = \begin{pmatrix} 0 \cdot 0 & 0 \cdot -2 & 0 \cdot 3 \\ -2 \cdot 0 & -2 \cdot -2 & -2 \cdot 3 \\ 4 \cdot 0 & 4 \cdot -2 & 4 \cdot 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -8 & 12 \end{pmatrix}$

$\mathbf{BA} = (0 \cdot 0 + -2 \cdot -2 + 3 \cdot 4) = (16) = 16$

Extra Observe that a 1×1 matrix is a scalar, just a simple number of variable.

Q8 Given are a matrix **A** of order 2×3 and a matrix **B** of order 3×3. Give the order of the result of the expression.

a. **AB**
b. **BA**
c. $(\mathbf{AA}')\mathbf{B}$
d. $(\mathbf{A}'\mathbf{A})\mathbf{B}$

A8 a. 2×3; b. not possible; c. not possible; d. 3×3

!!! Q9 (Sydsæter & Hammond, 4/E, 15.7.9)
A residential construction company plans to build several houses of three different types: 5 of type A, 7 of type B, and 12 of type C. Write down a 3-dimensional vector **x** whose coordinates give the umber of houses of each type. Suppose that each house of type A requires 20 units of timber, type B requires 18 units, and type C requires 25 units. Write down a vector **u** that gives the different timber quantities required for one house of each of the three different types A, B, and C. Find the total timber requirement by computing the inner product $\mathbf{u} \cdot \mathbf{x}$.

A9 $\mathbf{x} = \begin{pmatrix} 5 \\ 7 \\ 12 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 20 \\ 18 \\ 25 \end{pmatrix}$

$\mathbf{u} \cdot \mathbf{x} = u_1 x_1 + u_2 x_2 + u_3 x_3 = (5 \times 20) + (7 \times 18) + (12 \times 25) = 526$