Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Tutorial 4 – Full solutions

Instruction

In a tutorial session of 2 hours, we will obviously not be able to discuss all questions. Therefore, the following procedure applies:

- we expect students to prepare all exercises in advance;
- we will dicuss only a selection of exercises;
- exercises that were not discussed during class are nevertheless part of the course;
- students can indicate their wish list of exercises to be discussed during the session;
- teachers may invite students to answer questions, orally or on the blackboard.

!!! We further understand that your time is limited, and in particular that your time between lecture and tutorial may be limited. In case you have no time to prepare everything, we kindly advise you to give priority to the exercises that are indicated with the !!! sign.

Extreme values in two dimensions

!!! Q1 (Sydsæter & Hammond, 4/E, 13.1.1)
The function f defined for all (x, y) by $f(x, y) = -2x^2 - y^2 + 4x + 4y - 3$ has a maximum. Find the corresponding values of x and y.

A1 $(1,2)$

Sol $z = f(x, y) = -2x^2 - y^2 + 4x + 4y - 3 \Rightarrow$

$\frac{\partial z}{\partial x} = -4x + 4 = 0 \Rightarrow x = 1$

$\frac{\partial z}{\partial y} = -2y + 4 = 0 \Rightarrow y = 2$

So $(1,2)$ is the only stationary point of f. In the exercise it is given that this point $(1,2)$ is a maximum point.

!!! Q2 (based on Sydsæter & Hammond, 4/E, 13.1.3)
In a profit-maximizing problem, let the production function be given as
$Q = F(K, L) = 80 - (K - 3)^2 - 2(L - 6)^2 - (K - 3)(L - 6)$,
where Q is output, K is capital input, and L is labour input. The price per unit of output is $p = 1$, the cost (or rental) per unit of capital is $r = 0.65$, and the wage rate is $w = 1.2$. Find the only possible values of K and L that maximize profits.

A2 $(2.8,5.75)$

Sol $\pi(K, L) = pF(K, L) - rK - wL = pQ = \left(80 - (K - 3)^2 - 2(L - 6)^2 - (K - 3)(L - 6)\right) - 0.65K - 1.2L$

$\frac{\partial \pi}{\partial K} = -2(K - 3) - (L - 6) - 0.65 = 0$

$\frac{\partial \pi}{\partial L} = -4(L - 6) - (K - 3) - 1.2 = 0$

Solving this system of two linear equations gives: $K = 2.8$ and $L = 5.75$.
So the function $\pi = \pi(K, L)$ has only one stationary point: $(2.8,5.75)$. In the exercise it is given that this point $(2.8,5.75)$ is a maximum point.

Q3 (based on Sydsæter & Hammond, 4/E, 13.2.3)
Solve the utility-maximizing problem $\max U = xyz$ subject to $x + 3y + 4z = 108$, by making U a function of y and z by eliminating the variable x. Assume: $x > 0$, $y > 0$, and $z > 0$.

A3 $x = 36$, $y = 12$, $z = 9$, $U = 3888$

Sol Use the hint: $U = xyz = (108 - 3y - 4z)yz = 108yz - 3y^2z - 4yz^2$

$\frac{\partial U}{\partial y} = 108z - 6yz - 4z^2 = 0 \Rightarrow 2z(54 - 3y - 2z) = 0 \Rightarrow 54 - 3y - 2z = 0$

$\frac{\partial U}{\partial z} = 108y - 3y^2 - 8yz = 0 \Rightarrow y(108 - 3y - 8z) = 0 \Rightarrow 108 - 3y - 8z = 0$

Solving this system of two linear equations gives: $y = 12$, $z = 9$. So $x = 108 - 36 - 36 = 36$

In the exercise it is given that this poinr $(36,12,9)$ is a maximum point.

The maximum value is $U = 36 \times 12 \times 9 = 3888$

Extra Economic interpretation: xyz is the "utility" a person derives from consuming x, y, and z units, respectively, of three commodities. The prices per unit of the three commodities are 1, 3 and 4; income is 108.

Q4 (Sydsæter & Hammond, 4/E, 13.2.5)

A firm produces two goods. The cost of producing x units of good 1 and y units of good 2 is $C(x, y) = x^2 + xy + y^2 + x + y + 14$

Suppose the firm sells all its outputs of each good at prices per unit of p an q respectively. Find the values of x and y that maximize profits. (Assume $\frac{1}{2}p + \frac{1}{2} < q < 2p - 1$ and $p > 1$.)

A4 $\left(\frac{1}{3}(2p - q - 1), \frac{1}{3}(2q - p - 1)\right)$

Sol The profit is: $\pi = \pi(x, y) = px + qy - C(x, y) = -x^2 - xy - y^2 + (p - 1)x + (q - 1)y - 14$.

$\frac{\partial \pi}{\partial x} = -2x - y + p - 1 = 0 \Rightarrow 2x + y = p - 1$

$\frac{\partial \pi}{\partial y} = -x - 2y + q - 1 = 0 \Rightarrow x + 2y = q - 1$

Solving this system of two linear equations gives: $x = \frac{1}{3}(2p - q - 1)$, $y = \frac{1}{3}(2q - p - 1)$

Note: $x > 0 \Rightarrow \frac{1}{3}(2p - q - 1) > 0 \Rightarrow 2p - q - 1 > 0 \Rightarrow q < 2p - 1$

And: $y > 0 \Rightarrow \frac{1}{3}(2q - p - 1) > 0 \Rightarrow 2q - p - 1 > 0 \Rightarrow q > \frac{p+1}{2}$

So $\frac{p+1}{2} < q < 2p - 1$

This also implies $\frac{p+1}{2} < 2p - 1 \Rightarrow \frac{3}{2}p > \frac{3}{2} \Rightarrow \mathrm{p} > 1$

So the function $\pi = \pi(x, y)$ has only one stationary point: $\left(\frac{1}{3}(2p - q - 1), \frac{1}{3}(2q - p - 1)\right)$.

Furthermore, $\frac{\partial^2 \pi}{\partial x^2} = -2$, $\frac{\partial^2 \pi}{\partial y^2} = -2$, and $\frac{\partial^2 \pi}{\partial x \partial y} = -1$ so the second-order criterion gives:

$\frac{\partial^2 \pi}{\partial x^2} \cdot \frac{\partial^2 \pi}{\partial y^2} - \left(\frac{\partial^2 \pi}{\partial x \partial y}\right)^2 = 4 - 1 = 3 > 0 \Rightarrow$ in the stationary point $\left(\frac{1}{3}(2p - q - 1), \frac{1}{3}(2q - p - 1)\right)$ the profit π has an extreme value.

For determining if this stationary point is a minimum or a maximum we need:

$\left.\frac{\partial^2 \pi}{\partial x^2}\right]_{\left(\frac{1}{3}(2p-q-1), \frac{1}{3}(2q-p-1)\right)} = -2 < 0$ or $\left.\frac{\partial^2 \pi}{\partial y^2}\right]_{\left(\frac{1}{3}(2p-q-1), \frac{1}{3}(2q-p-1)\right)} = -2 < 0$

so the stationary point $\left(\frac{1}{3}(2p - q - 1), \frac{1}{3}(2q - p - 1)\right)$ is a maximum point.

Q5 (Sydsæter & Hammond, 4/E, 13.3.1)

(a) Find the partial derivatives of the first and second order for the function f defined for all (x, y) by $f(x, y) = 5 - x^2 + 6x - 2y^2 + 8y$.

(b) Find the only stationary point and classify it by using the second-order derivative test.

A5 $(3,2)$ is a maximum point

Sol (a): $\frac{\partial f}{\partial x} = -2x + 6$, $\frac{\partial f}{\partial y} = -4y + 8$, $\frac{\partial^2 f}{\partial x^2} = -2$, $\frac{\partial^2 f}{\partial y^2} = -4$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

(b): Solving the system of two linear equations $-2x+6=0$ and $-4y+8=0$ gives: $x=3$ and $y=2$, so the function $z=f(x,y)$ has only one stationary point: (3,2).

$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=8-0=8>0$ (for all stationary points)

so in the stationary point (3,2) the function f has an extreme value.

For determining if this stationary point is a minimum or a maximum we need $\left.\frac{\partial^2 z}{\partial x^2}\right]_{(3,2)}=-2<0$ or $\left.\frac{\partial^2 z}{\partial y^2}\right]_{(3,2)}=-4<0$. So the stationary point (3,2) is a (global) maximum point.

Q6 Compute the partial derivatives of the first and second order of $f(x,y)=x^3+2xy-5x-y^2$ and find the stationary points and classify them.

A6 (1,1) is a saddle point, and $\left(-\frac{5}{3},-\frac{5}{3}\right)$ is a maximum point.

Sol $z=f(x,y)=x^3+2xy-5x-y^2$.

$\frac{\partial z}{\partial x}=3x^2+2y-5$ and $\frac{\partial z}{\partial y}=2x-2y$

$\frac{\partial^2 z}{\partial x^2}=6x$, $\frac{\partial^2 z}{\partial y^2}=-2$, $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=2$

Stationary points: (1) $\frac{\partial z}{\partial x}=3x^2+2y-5=0$ and (2) $\frac{\partial z}{\partial y}=2x-2y=0$. (2) gives $x=y$, and substitution in (1) $3x^2+2x-5=0\Rightarrow x_1=1$ $(y_1=1)$ and $x_2=-\frac{5}{3}$ $\left(y_2=-\frac{5}{3}\right)$, so there are two stationary points: (1,1) and $\left(-\frac{5}{3},-\frac{5}{3}\right)$.

$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=-12x-4$

In the stationary point (1,1), $-12x-4]_{(1,1)}=-16<0$, so the function has a saddle point there.

In the stationary point $\left(-\frac{5}{3},-\frac{5}{3}\right)$, $-12x-4]_{\left(-\frac{5}{3},-\frac{5}{3}\right)}=16>0$, so the function has an extreme value there. For determining if this stationary point is a minimum or a maximum we need $\left.\frac{\partial^2 z}{\partial x^2}\right]_{\left(-\frac{5}{3},-\frac{5}{3}\right)}=6x]_{\left(-\frac{5}{3},-\frac{5}{3}\right)}=-10<0$ or (more easy:) $\left.\frac{\partial^2 z}{\partial y^2}\right]_{\left(-\frac{5}{3},-\frac{5}{3}\right)}=-2]_{\left(-\frac{5}{3},-\frac{5}{3}\right)}=-2<0$.

So the stationary point $\left(-\frac{5}{3},-\frac{5}{3}\right)$ is a maximum point.

!!! Q7 Consider the function f defined for all (x,y) by $f(x,y)=\frac{1}{2}x^2-x+ay(x-1)-\frac{1}{3}y^3+a^2y^2$, where a is a constant. Find all the stationary points of f and classify them.

A7 For $a\neq 0$, (1,0) is a minimum point and $(1-a^3,a^2)$ is a saddle point. For $a=0$, (1,0) is a saddle point.

Sol $z=f(x,y)=\frac{1}{2}x^2-x+ay(x-1)-\frac{1}{3}y^3+a^2y^2$

$\frac{\partial z}{\partial x}=x-1+ay=0\Rightarrow x=1-ay$ (1)

$\frac{\partial z}{\partial y}=a(x-1)-y^2+2a^2y=0$ (2)

Substitution of (1) in (2) gives: $-a^2y-y^2+2a^2y=0\Rightarrow a^2y-y^2=0\Rightarrow y(a^2-y)=0\Rightarrow y_1=0$ $(x_1=1)$ and $y_2=a^2$ $(x_2=1-a^3)$, so there are two stationary points: (1,0) and $(1-a^3,a^2)$.

Note: there is only one stationary point (1,0) in the case $a=0$.

For the second derivatives we find $\frac{\partial^2 z}{\partial x^2}=1$, $\frac{\partial^2 z}{\partial y^2}=-2y+2a^2$, and $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=a$.

$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=1\cdot(-2y+2a^2)-a^2=-2y+a^2$

In the stationary point $(1,0)$, $-2y + a^2]_{(1,0)} = a^2 > 0$ $(a \neq 0)$, so the function f has an extreme value. For determining if this stationary point is a minimum or a maximum we need $\frac{\partial^2 z}{\partial x^2}\Big]_{(1,0)} = 1 > 0$ or $\frac{\partial^2 z}{\partial y^2}\Big]_{(1,0)} = 2a^2 > 0$ $(a \neq 0)$. So stationary point $(1,0)$ is a minimum point in all cases except $a = 0$.
In the stationary point $(1 - a^3, a^2)$, $-2y + a^2]_{(1-a^3,a^2)} = -a^2 < 0$ $(a \neq 0)$, so the function f there has a saddle point.
In the case $a = 0$ there only is one stationary point $(1,0)$ but the test-criterion gives no answer. However, first note: $f(1,0) = -\frac{1}{2}$ and then that $f(1,y) = -\frac{1}{2} - \frac{1}{3}y^3$, so $f(1,y) < f(1,0)$ for all $y > 0$ and $f(1,y) > f(1,0)$ for all $y < 0$. Thus $(1,0)$ is neither a local maximum point, nor a local minimum point. Hence it is a saddle point.

Systems of linear equations

!!! Q1 (Sydsæter & Hammond, 4/E, 16.6.1.a)

Prove that $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}$

Sol $\mathbf{A}^{-1} = \mathbf{B} \Leftrightarrow \mathbf{AB} = \mathbf{I}$

$$\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} = \begin{pmatrix} \left(3 \times \frac{1}{3}\right) + \left(0 \times \frac{2}{3}\right) & (3 \times 0) + (0 \times -1) \\ \left(2 \times \frac{1}{3}\right) + \left(-1 \times \frac{2}{3}\right) & (2 \times 0) + (-1 \times -1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2 \text{ QED}$$

Extra Alternatively, you may work out $\begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}$

Q2 (Sydsæter & Hammond, 4/E, 16.6.1.b)

Prove that $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 8/7 & -1 & 3/7 \\ -2/7 & 0 & 1/7 \end{pmatrix}$

Sol $\mathbf{A}^{-1} = \mathbf{B} \Leftrightarrow \mathbf{AB} = \mathbf{I}$

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 8/7 & -1 & 3/7 \\ -2/7 & 0 & 1/7 \end{pmatrix} =$$

$$\begin{pmatrix} (1 \times -1) + \left(1 \times \frac{8}{7}\right) + \left(-3 \times -\frac{2}{7}\right) & (1 \times 1) + (1 \times -1) + (-3 \times 0) & (1 \times 0) + \left(1 \times \frac{3}{7}\right) + \left(-3 \times \frac{1}{7}\right) \\ (2 \times -1) + \left(1 \times \frac{8}{7}\right) + \left(-3 \times -\frac{2}{7}\right) & (2 \times 1) + (1 \times -1) + (-3 \times 0) & (2 \times 0) + \left(1 \times \frac{3}{7}\right) + \left(-3 \times \frac{1}{7}\right) \\ (2 \times -1) + \left(2 \times \frac{8}{7}\right) + \left(1 \times -\frac{2}{7}\right) & (2 \times 1) + (2 \times -1) + (1 \times 0) & (2 \times 0) + \left(2 \times \frac{3}{7}\right) + \left(1 \times \frac{1}{7}\right) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3 \text{ QED}$$

Q3 (Sydsæter & Hammond, 4/E, 16.6.4)

Let $\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$. Show that $\mathbf{A}^3 = \mathbf{I}_2$. Use this to find $\mathbf{A}^{-1}$.

A3 $\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$

Sol $\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A} = \frac{1}{4}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$

$\mathbf{A}^3 = \mathbf{A}^2\cdot\mathbf{A} = \frac{1}{4}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix} = \frac{1}{8}\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{I}_2$ QED

We can use the result $\mathbf{A}^3 = \mathbf{I}$ because

$\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{I}_2 = \mathbf{A}^3 \Rightarrow \mathbf{A}^{-1}\cdot\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{A}^{-1}\cdot\mathbf{A}^3 \Rightarrow \mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{I}_2 = \mathbf{A}^{-1}\cdot\mathbf{A}^3 = \mathbf{A}^2 =$
$\frac{1}{4}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$

Q4 The two 2×2-matrices $\mathbf{C}$ and $\mathbf{D}$ are given by: $\mathbf{C} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$. Further, $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Let X be an unknown 2×2-matrix. Solve $\mathbf{X}$ from the following matrix equation: $\mathbf{XC} = \mathbf{C} + \mathbf{D}^{-1}$.

A4 $\mathbf{X} = \begin{pmatrix}2 & -2\\ -3 & 8\end{pmatrix}$

Sol $\mathbf{XC} = \mathbf{C} + \mathbf{D}^{-1} \Rightarrow \mathbf{XCC}^{-1} = (\mathbf{C} + \mathbf{D}^{-1})\mathbf{C}^{-1} = \mathbf{CC}^{-1} + \mathbf{D}^{-1}\mathbf{C}^{-1} \Rightarrow \mathbf{XI}_2 = \mathbf{X} = \mathbf{I}_2 + \mathbf{D}^{-1}\mathbf{C}^{-1}$.

$\mathbf{X} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & -2\\ -3 & 7\end{pmatrix} = \begin{pmatrix}2 & -2\\ -3 & 8\end{pmatrix}$

!!! Q5 Let $\mathbf{B}$ be a 2×2-matrix: $\mathbf{B} = \begin{pmatrix}0 & \frac{1}{2}b\\ 1 & 0\end{pmatrix}$. For what value(s) of b does the matrix equation $\mathbf{B}^{-1} = \mathbf{B}$ hold?

A5 $b = 2$

Sol $\mathbf{B}^{-1} = \mathbf{B} \Leftrightarrow \mathbf{BB} = \mathbf{I}$. $\mathbf{B} = \begin{pmatrix}0 & \frac{1}{2}b\\ 1 & 0\end{pmatrix} \Rightarrow \mathbf{BB} = \begin{pmatrix}0 & \frac{1}{2}b\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}b\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}b & 0\\ 0 & \frac{1}{2}b\end{pmatrix} = \mathbf{I}_2$.

So $\frac{1}{2}b = 1 \Rightarrow b = 2$.

Q6 (based on Sydsæter & Hammond, 4/E, 15.6.1.b)

Verify that $\begin{pmatrix}-2 & 5 & 3\\ 3 & -3 & 0\\ 5 & 1 & -3\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix} = \begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}$.

Next use this result to solve the following system $\begin{cases}x_1 + 2x_2 + x_3 = 4\\ x_1 - x_2 + x_3 = 5\\ 2x_1 + 3x_2 - x_3 = 1\end{cases}$.

A6 $\mathbf{x} = \begin{pmatrix}\frac{20}{9}\\ -\frac{1}{3}\\ \frac{22}{9}\end{pmatrix}$

Sol $\begin{pmatrix}-2 & 5 & 3\\ 3 & -3 & 0\\ 5 & 1 & -3\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix} = \begin{pmatrix}-2+5+6 & -4-5+9 & -2+5-3\\ 3-3+0 & 6+3+0 & 3-3+0\\ 5+1-6 & 10-1-9 & 5+1+3\end{pmatrix} = \begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}$

QED

This means that $\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix}-2 & 5 & 3\\ 3 & -3 & 0\\ 5 & 1 & -3\end{pmatrix}$.

Rewrite the system of equations as $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} = \begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}4\\ 5\\ 1\end{pmatrix}$.

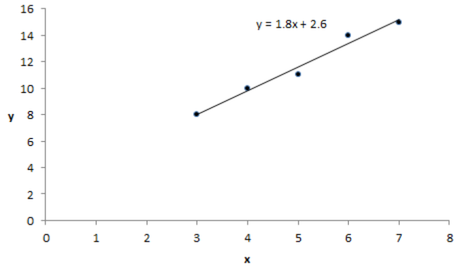
$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{9}\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -8+25+3 \\ 12-15+0 \\ 20+5-3 \end{pmatrix}\frac{1}{9}\begin{pmatrix} 20 \\ -3 \\ 22 \end{pmatrix} = \begin{pmatrix} \frac{20}{9} \\ -\frac{1}{3} \\ \frac{22}{9} \end{pmatrix}$$

Curve fitting

Q1 Given are five data points $(3,8)$, $(4,10)$, $(5,11)$, $(6,14)$, and $(7,15)$. Determine the optimal line that fits these point, using the least squares criterion. Tip: you can use Excel to speed up the calculations, but do not use the built-in trend-line, covariance, etc., but program the expression with sums.

A1 $y = 2.6 + 1.8x$

Sol Choose symbols, e.g., (x, y), and let the regression line be given as $y = a + bx$. Use the equations $b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$ and $a = \frac{1}{n}(\sum y - b\sum x)$ on the given data:
$x_1 = 3$, $x_2 = 4$, $x_3 = 5$, $x_6 = 16$, $x_5 = 15$, $y_1 = 8$, $y_2 = 10$, $y_3 = 11$, $y_4 = 14$, $y_5 = 15$ and $n = 5$.
This gives $n = 5$, $\sum x = 25$, $\sum y = 58$, $\sum x^2 = 135$, and $\sum xy = 308$. This yields $b = 1.8$ and $a = 2.6$. Therefore $y = 2.6 + 1.8x$ is the line that best fits the data with the least-squares criterion.
Indeed Excel estimates this trendline by the built-in function.



Extra Emphasize that $\sum x^2$ is not the same as $(\sum x)^2$, and that $\sum x \sum y$ is not the same as $\sum xy$.

!!! Q2 Given is the regresssion line $y = -15 + 210x$, where x is measured in km^2, and y is measured in euro.

(a) What happens with the regression coefficients -15 and 210 when we decide to measure y in keuro instead of in euro?

(b) And what happens if we measure x in m^2 instead of km^2?

(c) Develop a general formula for what happens with regession coefficient a and b in the regression line $y = a + bx$, when x is changed into αx and y is changed into βy.

A2 (a) a and b scale down by a factor 1000 (b) a remains the same, and b scales down by a factor 1,000,000 (c) a scales up by a factor β, and b scales up by a factor $\frac{\beta}{\alpha}$.

Sol

(a) The unit of variable y changes from euro to keuro, so $y' = \frac{1}{1000}y$.
The formula for y' then becomes $y' = a' + b'x$. Therefore $\frac{1}{1000}y = a' + b'x$.
Substituting $y = a + bx$ we find $\frac{1}{1000}(a + bx) = a' + b'x$. This holds for every pair of points (x_i, y_i).
This implies that $a' = \frac{1}{1000}a$ and $b' = \frac{1}{1000}b$. So both regression coefficients scale down by a factor 1000.

(b) The unit of variable x changes from km² to m², so $x' = 10^6 x$.
The formula for y then becomes $y = a' + b'x'$. Therefore $y = a' + 10^6 b'x$.
Substituting $y = a + bx$ we find $a + bx = a' + 10^6 b'x$. This holds for every pair of points (x_i, y_i).
This implies that $a' = a$ and $b' = \frac{1}{10^6} b = 10^{-6} b$. So regression coefficient a remains the same, and regression coefficient b scales down by a factor 1,000,000.

(c) The formula for y becomes $y' = a' + b'x'$. Therefore $\beta y = a' + b'\alpha x$.
Substituting $y = a + bx$ we find $\beta(a + bx) = a' + b'\alpha x$. This holds for every pair of points (x_i, y_i).
This implies that $a' = \beta a$ and $b' = \frac{\beta}{\alpha} b$. So regression coefficient a scales up by a factor β, and regression coefficient b scales up by a factor $\frac{\beta}{\alpha}$.

Q3 A data set with $n = 100$ gives a regresssion line $y = 15 + 210x$. Which statements are true?
(A) There are points with a negative y-value.
(B) The covariance of x and y is positive.
(C) The mean of y is bigger than the mean of x.
(D) Almost all data points are in the first or third qudrant.
(E) None of the above statements is true.
A3 (B) is true.
Sol (A) is not true, you don't know anything about the location of the values. (B) is true: a positive slope (here 210) implies a positive covariance. (C) is not necessarily true: although $\bar{y} = 210\bar{x} + 15$, $\bar{y} > \bar{x}$ only when $\bar{x} > 0$, but it may be the case that $\bar{x} < 0$. (D) it is tempting to believe this is true, however, it isn't. You may easily sketch a counterexample.

Q4 Does the OLS procedure always produce a best line estimate? Interpret any exception.
A4 No, not when the regression line is vertical.
Sol The denominator of b requires that $n\sum x^2 - (\sum x)^2 \neq 0$. This implies that $\sigma_x^2 \neq 0$. When $\sigma_x^2 = 0$, the slope will be infinity, the regression line will be vertical.

Q5 Consider again the five data points (3,8), (4,10), (5,11), (6,14), and (7,15). Write the regression problem in matrix form.

A5
$$\begin{pmatrix} 8 \\ 10 \\ 11 \\ 14 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} e_1 \\ e_1 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

Sol With $\mathbf{X} = (\mathbf{1} \quad \mathbf{x}) = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 8 \\ 10 \\ 11 \\ 14 \\ 15 \end{pmatrix}$ we have $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$, where $\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix}$.

So we have
$$\begin{pmatrix} 8 \\ 10 \\ 11 \\ 14 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} e_1 \\ e_1 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

Gaussian elimination

!!! Q1 (Sydsæter & Hammond, 4/E, 15.6.1.b)

Solve the following system by Gaussian elimination: $\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_1 - x_2 + x_3 = 5 \\ 2x_1 + 3x_2 - x_3 = 1 \end{cases}$.

A1 $\begin{cases} x_1 = \frac{20}{9} \\ x_2 = -\frac{1}{3} \\ x_3 = \frac{22}{9} \end{cases}$

Sol
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 5 \\ 2 & 3 & -1 & 1 \end{array}\right) \xrightarrow{r_2 - r_1, r_3 - 2r_1}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & -3 & -7 \end{array}\right) \xrightarrow{r_2 \leftrightarrow r_3}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & -3 & -7 \\ 0 & -3 & 0 & 1 \end{array}\right) \xrightarrow{r_3 - 3r_2}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & -3 & -7 \\ 0 & 0 & 9 & 22 \end{array}\right) \xrightarrow{r_1 \times 9, r_2 \times 3}$$
$$\left(\begin{array}{ccc|c} 9 & 18 & 9 & 36 \\ 0 & -3 & -9 & -21 \\ 0 & 0 & 9 & 22 \end{array}\right) \xrightarrow{r_1 - r_3, r_2 + r_3}$$
$$\left(\begin{array}{ccc|c} 9 & 18 & 0 & 14 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 9 & 22 \end{array}\right) \xrightarrow{r_1 + 6r_2}$$
$$\left(\begin{array}{ccc|c} 9 & 0 & 0 & 20 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 9 & 22 \end{array}\right) \Rightarrow \begin{cases} x_1 = \frac{20}{9} \\ x_2 = -\frac{1}{3} \\ x_3 = \frac{22}{9} \end{cases}$$

!!! Q2 (Sydsæter & Hammond, 4/E, 15.6.1.c)

Solve the following system by Gaussian elimination: $\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$.

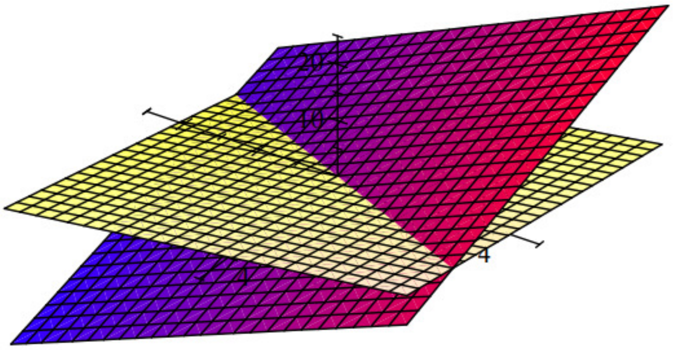
A2 $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\lambda \\ \frac{3}{5}\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$

Sol
$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -3 & 1 & 0 \end{array}\right) \xrightarrow{r_2 - 2r_1}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right) \xrightarrow{r_1 \times 5}$$
$$\left(\begin{array}{ccc|c} 5 & 5 & -5 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right) \xrightarrow{r_1 + r_2}$$
$$\left(\begin{array}{ccc|c} 5 & 0 & -2 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right) \Rightarrow \text{more solutions! Choose } x_3 = \lambda \Rightarrow$$
$$\begin{cases} 5x_1 - 2\lambda = 0 \\ -5x_2 + 3\lambda = 0 \end{cases} \Rightarrow \begin{cases} 5x_1 = 2\lambda \\ 5x_2 = 3\lambda \end{cases} \Rightarrow \begin{cases} x_1 = \frac{2}{5}\lambda \\ x_2 = \frac{3}{5}\lambda \end{cases} \Rightarrow$$

general solution is: $(x_1, x_2, x_3) = \left(\frac{2}{5}\lambda, \frac{3}{5}\lambda, \lambda\right)$ for any real number λ.

The set of solutions is $\left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\lambda \\ \frac{3}{5}\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R}\right\}$

Extra You may visualize the result as two intersecting planes (defined by the two equations), their intersection is clearly a line in 3-dimensional space.



Q3 (Sydsæter & Hammond, 4/E, 15.6.2)
Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b: $\begin{cases} x + y - z = 1 \\ x - y + 2z = 2 \\ x + 2y + az = b \end{cases}$

A3 1 solution: $a \neq -\frac{5}{2}$; no solutions: $a = -\frac{5}{2}$ and $b \neq \frac{1}{2}$; more solutions: $a = -\frac{5}{2}$ and $b = \frac{1}{2}$

Sol
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 2 & a & b \end{array}\right) \xrightarrow{r_2 - r_1, r_3 - r_1}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 1 & a+1 & b-1 \end{array}\right) \xrightarrow{r_2 \leftrightarrow r_3}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+1 & b-1 \\ 0 & -2 & 3 & 1 \end{array}\right) \xrightarrow{r_3 + 2r_2}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+1 & b-1 \\ 0 & 0 & 2a+5 & 2b-1 \end{array}\right)$$

Conclusion: see third row!

type $0 \quad 0 \quad \neq 0 \quad | \cdots$: 1 solution: $2a + 5 \neq 0 \Rightarrow a \neq -\frac{5}{2}$

type $0 \quad 0 \quad 0 \quad | \neq 0$: no solutions: $a = -\frac{5}{2}$ and $b \neq \frac{1}{2}$

type $0 \quad 0 \quad 0 \quad | 0$: more solutions: $a = -\frac{5}{2}$ and $b = \frac{1}{2}$

Extra So far, we only discussed the number of solutions. The system can also be solved for the various choices of a and b.

Case 1. $a \neq -\frac{5}{2}$ (one solution)

$(2a + 5)z = 2b - 1 \Rightarrow z = \frac{2b-1}{2a+5}$.

Substitute in second equation: $y + (a+1)z = b - 1 \Rightarrow y = b - 1 - (a+1)\frac{2b-1}{2a+5} = \frac{-a+3b-4}{2a+5} \Rightarrow$
$x + y - z = 1 \Rightarrow x = 1 - y + z = 1 - \frac{-a+3b-4}{2a+5} + \frac{2b-1}{2a+5} = \frac{3a-b+8}{2a+5}$.

Case 2. $a = -\frac{5}{2}$ and $b \neq \frac{1}{2}$ (no solutions)

Case 3. $a = -\frac{5}{2}$ and $b = \frac{1}{2}$ (more solutions)

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Choose $z = \lambda \Rightarrow x + \frac{1}{2}\lambda = \frac{3}{2} \Rightarrow x = \frac{3}{2} - \frac{1}{2}\lambda$ and $y - \frac{3}{2}\lambda = -\frac{1}{2} \Rightarrow y = -\frac{1}{2} + \frac{3}{2}\lambda$

General solution is $(x, y, z) = \left(\frac{3}{2} - \frac{1}{2}\lambda, -\frac{1}{2} + \frac{3}{2}\lambda, \lambda\right)$ for any real number λ.

The set of solutions is $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{1}{2}\lambda \\ -\frac{1}{2} + \frac{3}{2}\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$

Q4 (Sydsæter & Hammond, 4/E, 15.Review.9)
Use the Gaussian elimination to find for what values of a the following system has solutions. Then find all the possible solutions.

$$\begin{cases} x + ay + 2z = 0 \\ -2x - ay + z = 4 \\ 2ax + 3a^2y + 9z = 4 \end{cases}$$

A4 $a = 1$: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\lambda - 4 \\ 4 - 5\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$; $a = 0$: no solution; $a \neq 1$ and $a \neq 0$: one solution: $x = -\frac{8}{3}$, $y = \frac{16}{9a}$, $z = \frac{4}{9}$.

Sol
$$\begin{pmatrix} 1 & a & 2 & 0 \\ -2 & -a & 1 & 4 \\ 2a & 3a^2 & 9 & 4 \end{pmatrix} \xrightarrow{r_2 + 2r_1, r_3 - 2ar_1}$$

$$\begin{pmatrix} 1 & a & 2 & 0 \\ 0 & a & 5 & 4 \\ 0 & a^2 & 9 - 4a & 4 \end{pmatrix} \xrightarrow{r_1 - r_2, r_3 - ar_2}$$

$$\begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & a & 5 & 4 \\ 0 & 0 & 9 - 9a & 4 - 4a \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & a & 5 & 4 \\ 0 & 0 & 9(1 - a) & 4(1 - a) \end{pmatrix}$$

Distinguish:

Case 1. $a = 1$: $\begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & a & 5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 5 & 4 \end{pmatrix} \Rightarrow$ more solutions!

Choose: $z = \lambda \Rightarrow \begin{cases} x - 3\lambda = -4 \\ y + 5\lambda = 4 \end{cases} \Rightarrow \begin{cases} x = 3\lambda - 4 \\ y = 4 - 5\lambda \end{cases}$

The set of solutions is: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\lambda - 4 \\ 4 - 5\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$

Case 2. $a = 0$: $\begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 9 & 4 \end{pmatrix} \xrightarrow{r_3 - \frac{9}{5}r_2} \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & -\frac{16}{5} \end{pmatrix} \Rightarrow$ in this case there is no solution.

Case 3. $a \neq 1$ and $a \neq 0$: $\begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & a & 5 & 4 \\ 0 & 0 & 9(1 - a) & 4(1 - a) \end{pmatrix} \xrightarrow{r_3 \times \frac{1}{1-a}} \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & a & 5 & 4 \\ 0 & 0 & 9 & 4 \end{pmatrix} \Rightarrow$

$9z = 4 \Rightarrow z = \frac{4}{9}$

$ay + 5z = 4 \Rightarrow ay = 4 - 5z = 4 - \frac{20}{9} = \frac{16}{9} \Rightarrow y = \frac{16}{9a}$

$x - 3z = -4 \Rightarrow x = 3z - 4 = \frac{12}{9} - 4 = -\frac{24}{9} = -\frac{8}{3}$

In all these cases there is one solution: $x = -\frac{8}{3}$, $y = \frac{16}{9a}$, $z = \frac{4}{9}$.

!!! Q5 (based on Sydsæter & Hammond, 4/E, 16.7.5.a)

Use Gauss-Jordan elimination to calculate the inverse (provided it exists) of the following matrix: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

A5 $\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ 1\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

Sol $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{r_2 - 3r_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{r_1 + r_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{r_2/2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} \end{array}\right) \Rightarrow$

$\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ 1\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

!!! Q6 Let $\mathbf{A}$ be a 2×2-matrix: $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 4 & a \end{pmatrix}$. For what value(s) of a does the inverse matrix $\mathbf{A}^{-1}$ not exist?

A6 $a = 12$

Sol $\left(\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 4 & a & 0 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1} \left(\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 0 & a - 12 & -2 & 1 \end{array}\right)$. $\mathbf{A}$ has inverse $\Leftrightarrow a - 12 \neq 0 \Leftrightarrow a \neq 12$. So, when $a = 12$, $\mathbf{A}$ has no inverse.

Extra Matrix on the right-hand side is not needed or the conclusion.