Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Tutorial 5 – Full solutions

Instruction

In a tutorial session of 2 hours, we will obviously not be able to discuss all questions. Therefore, the following procedure applies:

- we expect students to prepare all exercises in advance;
- we will dicuss only a selection of exercises;
- exercises that were not discussed during class are nevertheless part of the course;
- students can indicate their wish list of exercises to be discussed during the session;
- teachers may invite students to answer questions, orally or on the blackboard.

!!! We further understand that your time is limited, and in particular that your time between lecture and tutorial may be limited. In case you have no time to prepare everything, we kindly advise you to give priority to the exercises that are indicated with the !!! sign.

Implicit differentiation

!!! Q1 (Sydsæter & Hammond, 4/E, 7.1.2)
For the equation $x^2y = 1$, find dy/dx and d^2y/dx^2 by implicit differentiation. Check by solving the equation for y and then differentiating.

A1 $y' = \frac{-2y}{x}$

$y'' = \frac{-(2y+4xy')}{x^2}$

Sol $x^2y = 1$ (note: of course $x \neq 0$, $y(x) = 0$)

$\Rightarrow 2xy + x^2y' = 0 \Rightarrow x^2y' = -2xy \Rightarrow y' = \frac{-2xy}{x^2} = \frac{-2y}{x}$

By solving for y first: $y = \frac{1}{x^2} = x^{-2} \Rightarrow \frac{dy}{dx} = y' = -2x^{-3} = \frac{-2}{x^3}$. Because $y = \frac{1}{x^2}$ both expressions for y' are the same.

For the second derivative use: $2xy + x^2y' = 0$.

$\Rightarrow 2y + 2xy' + 2xy' + x^2y'' = 2y + 4xy' + x^2y'' = 0 \Rightarrow x^2y'' = -(2y + 4xy') \Rightarrow y'' = \frac{-(2y+4xy')}{x^2}$

Here also possible: $\frac{d^2y}{dx^2} = y'' = 6x^{-4} = \frac{6}{x^4}$. Both expressions for y'' are the same (using again $y = \frac{1}{x^2}$).

Q2 (Sydsæter & Hammond, 4/E, 7.1.5)
Suppose that y is a differentiable function of x that satisfies the equation

$$2x^2 + 6xy + y^2 = 18$$

Find y' and y''at the point $(x, y) = (1,2)$.

A2 $y'|_{(1,2)} = -\frac{8}{5}$

$y''|_{(1,2)} = \frac{126}{125}$

Sol $2x^2 + 6xy + y^2 = 18 \Rightarrow 4x + 6y + 6xy' + 2yy' = 0 \Rightarrow 6xy' + 2yy' = -(4x + 6y) \Rightarrow$
$(6x + 2y)y' = -(4x + 6y) \Rightarrow y' = \frac{-(4x+6y)}{(6x+2y)} \Rightarrow y'|_{(1,2)} = \frac{-16}{10} = -\frac{8}{5}$

$4x + 6y + 6xy' + 2yy' = 0 \Rightarrow 4 + 6y' + 6xy'' + 2y'y' + 2yy'' = 0 \Rightarrow (6x + 2y)y'' =$
$-[4 + 12y' + 2(y')^2] \Rightarrow y'' = \frac{-[4+12y'+2(y')^2]}{(6x+2y)} \Rightarrow y''|_{(1,2)} = \frac{\frac{252}{25}}{10} = \frac{252}{250} = \frac{126}{125}$

!!! Q3 (based on Sydsæter & Hammond, 4/E, 7.4.8)

(a) The equation

$$2x^2 + 6xy + y^2 = 18$$

defines y as a differentiable function of x about the point $(x, y) = (1,2)$. Find the slope of the graph at this point by implicit differentiation.

(b) What is the linear approximation about $x = 1$?

A3 (a) $y'|_{(1,2)} = -\frac{8}{5} \Rightarrow y(x) \approx 2 - \frac{8}{5}(x - 1)$

Sol (a) Remember from Q2 for point (1,2): $y'|_{(1,2)} = -\frac{8}{5}$

(b) $y(x) \approx y|_{(1,2)} + y'|_{(1,2)} \frac{(x-1)}{1!} = 2 - \frac{8}{5}(x - 1)$ $(= -\frac{8}{5}x + \frac{18}{5})$

Numbers and units

!!! Q1 You combine data x and y from different sources, using different notations. State the value of $z = xy$.

(a) $x = 4$ million, $y = 3\text{E}7$.

(b) $x = 3 \times 10^{-5}$, $y = 2\text{E}6$.

(c) $x = 1{,}200$, $y = 0.000{,}1$.

(d) $x = 4\text{E} - 5$, $y = 3$ thousandth

A1 (a) $1.2 \cdot 10^{14}$ (b) 60 (c) 0.12 (d) $1.2 \cdot 10^{-7}$

Sol (a) $z = xy = 4 \cdot 10^6 \times 3 \cdot 10^7 = 4 \times 3 \cdot 10^{6+7} = 12 \cdot 10^{13} = 1.2 \cdot 10^{14}$

(b) $z = xy = 3 \cdot 10^{-5} \times 2 \cdot 10^6 = 3 \times 2 \cdot 10^{-5+6} = 6 \cdot 10^1 (= 60)$

(c) $z = xy = 1.2 \cdot 10^3 \times 1 \cdot 10^{-4} = 1.2 \times 1 \cdot 10^{3-4} = 1.2 \cdot 10^{-1} = (0.12)$

(d) $z = xy = 4 \cdot 10^{-5} \times 3 \cdot 10^{-3} = 4 \times 3 \cdot 10^{-5-3} = 12 \cdot 10^{-8} = 1.2 \cdot 10^{-7}$

Extra In (c) you're not fully sure what 1,200 means.

Q2 You combine data with different units. State the value and unit of z.

(a) $x = 5$ km, $y = 3$ hr, $z = xy$.

(b) $x = 8$ gallon, $y = 2$ inch^2, $z = \sqrt{\frac{x}{y}}$

(c) $x = -3$ euro, $y = 4$ kg, $z = x + y$

(d) $K = 4$ B\$, $L = 3$ B\$, $z = K^{0.2} L^{0.8}$

A2 (a) 15 km $\cdot$ hr (b) $2 \frac{\sqrt{\text{gallon}}}{\text{inch}}$ (c) impossible (d) 3.2 B\$

Sol (a) $z = (15\text{ km}) \cdot (3\text{ hr}) = (15 \cdot 3)(\text{km} \cdot \text{hr}) = 15\text{ km} \cdot \text{hr}$

(b) $z = \sqrt{\frac{8\text{ gallon}}{2\text{ inch}^2}} = \sqrt{\frac{8}{2}}\sqrt{\frac{\text{gallon}}{\text{inch}^2}} = 2\sqrt{\frac{\text{gallon}}{\text{inch}^2}} = 2\frac{\sqrt{\text{gallon}}}{\text{inch}}$

(c) $z = -3$ euro $+ 4$ kg, impossible!

(d) $z = (4\text{ B\$})^{0.2}(3\text{ B\$})^{0.8} = (4^{0.2} \cdot 3^{0.8})(\text{B\$}^{0.2} \cdot \text{B\$}^{0.8}) = 3.2\,(\text{B\$}^{0.2+0.8}) = 3.2\text{ B\$}$

Extra Remark that the exponents (here 0.2 and 0.8) must add up to 1 to form a meaningful unit. B\$ means billion \$.

!!! Q3 You combine data with different units. State the value and unit of z.

(a) $x = 300$ mm, $y = 0.5$ cm, $z = x + y$

(b) $x = 12\text{E}6$ s, $y = 6$ month, $z = x + y$

(c) $x = 50$ keuro, $w = 50$ \$/day, $T = 1$ yr, 1 \$ $= 0.8$ euro, $z = x + wT$

A3 (a) 305 mm (b) $27 \cdot 10^6$ s (c) $65 \cdot 10^3$ euro

Sol (a) $z = 3 \cdot 10^{-1}\text{ m} + 5 \cdot 10^{-3}\text{ m} = (300 + 5) \cdot 10^{-3}\text{ m} = 305 \cdot 10^{-3}\text{ m} (= 305\text{ mm})$

(b) $z = (12 \cdot 10^6 \text{ s}) + (6 \times \text{ month}) = (12 \cdot 10^6 \text{ s}) + \left(6 \text{ month} \times 30 \frac{\text{day}}{\text{month}} \times 24 \frac{\text{hr}}{\text{day}} \times 3600 \frac{\text{s}}{\text{hr}}\right) = (12 \cdot 10^6 \text{ s}) + \left((6 \cdot 30 \cdot 24 \cdot 3600) \text{ s}\right) = (12 \cdot 10^6 + 15{,}552{,}000) \text{ s} = 27 \cdot 10^6 \text{ s}$

(c) $z = 50 \cdot 10^3 \text{ euro} + (50\ \$ \cdot \text{day}^{-1}) \times (1 \text{ yr}) = 50 \cdot 10^3 \text{ euro} + (50 \times 0.8 \text{ euro} \cdot \text{day}^{-1}) \times \left(1 \text{ yr} \times 365 \frac{\text{day}}{\text{yr}}\right) = 50 \cdot 10^3 \text{ euro} + 50 \times 0.8 \times 365 \text{ euro} = 50 \cdot 10^3 \text{ euro} + 14600 \text{ euro} = 65 \cdot 10^3 \text{ euro}$

Extra Remark that the time units are a bit imprecisely defined. Are there 365 days in a year, 366, or something in between? How many days in one month? That is also a reason to keep the number of significant digit limited in (b) and (c).

Constrained optimization

!!! Q1 (Sydsæter & Hammond, 4/E, 14.1.1.a)
Use Lagrange's method to find the only possible solution to the problem:
$\max xy$ subject to $x + 3y = 24$.

A1 $(12{,}4)$ with $\lambda = 4$

Sol The Lagrangian is $\mathcal{L}(x, y, \lambda) = xy - \lambda(x + 3y - 24) \Rightarrow$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = x - 3\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -x - 3y + 24 = 0 \end{cases} \Rightarrow \begin{cases} y = \lambda \\ x = 3\lambda \\ x + 3y = 24 \end{cases}$$

Substitution of the first two equations in the third equation gives $3\lambda + 3\lambda = 24 \Rightarrow \lambda = 4$

So $\begin{cases} x = 12 \\ y = 4 \\ \lambda = 4 \end{cases}$. So the only solution candidate to the constrained optimization problem is $(12{,}4)$ with $\lambda = 4$. This point gives a maximum.

Q2 (Sydsæter & Hammond, 4/E, 14.3.1.b)
$\max (\min)\ x + y$ subject to $x^2 + 3xy + 3y^2 = 3$

A2 $f(-3{,}1) = -2$ (minimum), $f(3, -1) = 2$ (maximum)

Sol The Lagrangian is: $\mathcal{L}(x, y, \lambda) = x + y - \lambda(x^2 + 3xy + 3y^2 - 3)$

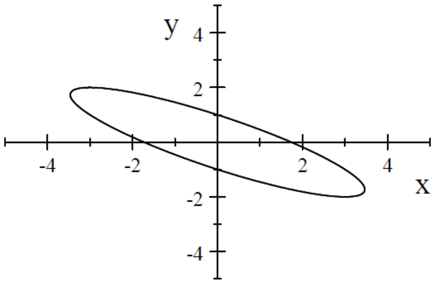
So $$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x - 3\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 1 - 3\lambda x - 6\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -x^2 - 3xy - 3y^2 + 3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda(2x + 3y) = 1 \\ \lambda(3x + 6y) = 1 \\ x^2 + 3xy + 3y^2 = 3 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{1}{2x+3y} \\ \lambda = \frac{1}{3x+6y} \\ x^2 + 3xy + 3y^2 = 3 \end{cases} \Rightarrow$$

$\frac{1}{2x+3y} = \frac{1}{3x+6y} \Rightarrow 3x + 6y = 2x + 3y \Rightarrow x = -3y \quad (2x + 3y \neq 0 \text{ and } 3x + 6y \neq 0)$

Substitution gives $9y^2 - 9y^2 + 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y_1 = 1, y_2 = -1 \Rightarrow x_1 = -3, x_2 = 3$

This gives two points: $\left(-3{,}1, -\frac{1}{3}\right)$ and $\left(3, -1, \frac{1}{3}\right)$.

These two points are the only possible solutions to the max/min problem, respectively. There now are several solution candidates to the constrained optimization problem (use the figure of the constraint $x^2 + 3xy + 3y^2 = 3$!).



We pick the one which gives the largest (or smallest) value to the objective function:
$\left(-3,1,-\frac{1}{3}\right)$: $f(-3,1) = x + y]_{(-3,1)} = -3 + 1 = -2$ (minimum)
$\left(3,-1,\frac{1}{3}\right)$: $f(3,-1) = x + y]_{(3,-1)} = 3 - 1 = 2$ (maximum)

Q3 (Sydsæter & Hammond, 4/E, 14.3.1.a)
max (min) $3xy$ subject to $x^2 + y^2 = 8$

A3 $f(2,2) = f(-2,-2) = 12$ (maximum), $f(-2,2) = f(2,-2) = -12$ (minimum)

Sol The Lagrangian is: $\mathcal{L}(x, y, \lambda) = 3xy - \lambda(x^2 + y^2 - 8)$

So $\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 3y - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 3x - 2\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -x^2 - y^2 + 8 = 0 \end{cases} \Rightarrow \begin{cases} 2\lambda x = 3y \\ 2\lambda y = 3x \\ x^2 + y^2 = 8 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{3y}{2x} \\ \lambda = \frac{3x}{2y} \\ x^2 + y^2 = 8 \end{cases} \Rightarrow$

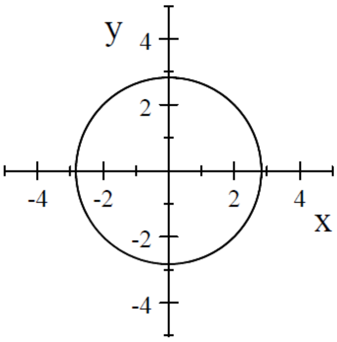
$\frac{3y}{2x} = \frac{3x}{2y} \Rightarrow 6y^2 = 6x^2 \Rightarrow x^2 = y^2 \Rightarrow x = y$ or $x = -y$

Substitution gives in both cases $2y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y_1 = 2, y_2 = -2 \Rightarrow x_1 = 2, x_2 = -2$

For $y_1 = 2$, we find two points: $\left(2,2,\frac{3}{2}\right)$ and $\left(-2,2,-\frac{3}{2}\right)$

For $y_1 = -2$, we find two other points: $\left(-2,-2,\frac{3}{2}\right)$ and $\left(2,-2,-\frac{3}{2}\right)$.

These four points are the only possible solutions to the max/min problems, respectively.
There now are several solution candidates to the constrained optimization problem (use the figure of the constraint $x^2 + y^2 = 8$!). We pick the one which gives the largest (or smallest) value to the objective function:



$(2,2) \Rightarrow f(2,2) = 3xy]_{(2,2)} = 12$ (maximum)
$(-2,2) \Rightarrow f(-2,2) = 3xy]_{(-2,2)} = -12$ (minimum)
$(-2,-2) \Rightarrow f(-2,-2) = 3xy]_{(-2,-2)} = 12$ (maximum)
$(2,-2) \Rightarrow f(2,-2) = 3xy]_{(2,-2)} = -12$ (minimum)

So $(2,2)$ and $(-2,-2)$ are the only possible solutions of the maximization problem, and $(-2,2)$ and $(2,-2)$ are the only possible solutions of the minimization problem.

!!! Q4 A manufacturer wants to produce cylindrical cans with a content of exactly 1 liter using as few tin as possible. So the total outside surface of a can has to be minimal. Determine the height h and the radius r of upper and bottom surface of such a can (in centimeters).
Note: The surface of the top side of the can (and the bottom of the can) can be calculated by πr^2 squared centimeters (cm^2); calculation of the surface of the side of the can gives $2\pi r h$ cm^2. The content of the can has to be $\pi r^2 h = 1000$ cm^3.

A4 $h = 2\sqrt[3]{\frac{500}{\pi}},\ r = \sqrt[3]{\frac{500}{\pi}}$

Sol The problem is: minimize $f(h,r) = 2\pi r^2 + 2\pi r h$ subject to $g(h.r) = \pi r^2 h = 1000$ ($h > 0$, $r > 0$).
The Lagrangian is $\mathcal{L}(h,r,\lambda) = 2\pi r^2 + 2\pi r h - \lambda(\pi r^2 h - 1000)$.

$$\text{So} \begin{cases} \frac{\partial \mathcal{L}}{\partial h} = 2\pi r - \lambda \pi r^2 = 0 \\ \frac{\partial \mathcal{L}}{\partial r} = 4\pi r + 2\pi h - 2\lambda \pi r h = 0 \\ \frac{\partial \mathcal{L}}{\partial h} = -\pi r^2 h + 1000 \end{cases} \Rightarrow \begin{cases} \lambda r = 2 \\ \lambda h r = 2r + h \\ \pi r^2 h = 1000 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{2}{r} \\ \lambda = \frac{2r+h}{hr} \\ \pi r^2 h = 1000 \end{cases} \Rightarrow$$

$$\frac{2}{r} = \frac{2r+h}{hr} \Rightarrow 2hr = 2r^2 + hr \Rightarrow hr = 2r^2 \Rightarrow h = 2r$$

Substitution gives $2\pi r^3 = 1000 \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$

This implies $(h,r) = \left(2\sqrt[3]{\frac{500}{\pi}}, \sqrt[3]{\frac{500}{\pi}}\right)$ so the height and the diameter of the can have to be the same: $h = 2r = 2\sqrt[3]{\frac{500}{\pi}}$ (≈ 10.8385)

Extra Another method is also possible here.
Substitute $h = \frac{1000}{\pi r^2}$ in $f(h,r) = f^*(r) = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$ and minimize $f^*(r)$:

$$\frac{df^*}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow 4\pi r = \frac{2000}{r^2} \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}} \text{ and } h = \frac{1000}{\pi}\left(\frac{\pi}{500}\right)^{\frac{2}{3}} = 2\sqrt[3]{\frac{500}{\pi}}$$

Applications of integrals

!!! Q1 (Sydsæter & Hammond, 4/E, 9.2.2.d)
Compute the area bounded by the graph of the function over the indicated interval: $f(x) = 1/x^2$ in $[1,10]$

A1 $\frac{9}{10}$

Sol Note $f(x) = 1/x^2 > 0$ for all $x \in [1,10]$ so the area can be computed by $\int_1^{10} \frac{1}{x^2} dx$.

$$\int_1^{10} \frac{1}{x^2} dx = \int_1^{10} x^{-2} dx = [-x^{-1}]_1^{10} = \left[-\frac{1}{x}\right]_1^{10} = -\frac{1}{10} - (-1) = \frac{9}{10}$$

Extra Emphasize the check that the function is non-negative on the entire interval.

Q2 (Sydsæter & Hammond, 4/E, 9.2.4)
Compute the area A bounded by the graph of $f(x) = \frac{1}{2}(e^x + e^{-x})$, the x-axis, and the lines $x = -1$ and $x = 1$.

A2 $e - e^{-1}$

Sol Note: $f(x) = \frac{1}{2}(e^x + e^{-x}) > 0$ for all x so the area can be computed by $\int_{-1}^{1} \frac{1}{2}(e^x + e^{-x}) dx$.

$\int_{-1}^{1}\frac{1}{2}(e^x+e^{-x})dx = \frac{1}{2}\int_{-1}^{1}(e^x+e^{-x})dx = \frac{1}{2}[e^x-e^{-x}]_{-1}^{1} = \frac{1}{2}\left(e-e^{-1}-(e^{-1}-e)\right) = e-e^{-1}$

!!! Q3 (Sydsæter & Hammond, 4/E, 9.4.7)
Suppose the demand and supply curves are $P=f(Q)=\frac{6000}{Q+50}$ and $P=g(Q)=Q+10$. Find the equilibrium price and quantity, and compute the consumer and producer surplus.

A3 $Q^*=50$, $P^*=60$, $CS=6000\ln 2-3000$, $PS=1250$

Sol The equilibrium quantity Q^* follows from $\frac{6000}{Q^*+50}=Q^*+10 \Rightarrow (Q^*+50)(Q^*+10)=6000 \Rightarrow Q^{*2}+60Q^*+500=6000 \Rightarrow Q^{*2}+60Q^*-5500=0 \Rightarrow (Q^*+110)(Q^*-50)=0 \Rightarrow Q^*=50 \Rightarrow P^*=60$ (note: $Q^*=-110$ is impossible)

$CS=\int_0^{Q^*}(f(Q)-P^*)dQ=\int_0^{50}\left(\frac{6000}{Q+50}-P^*\right)dQ=[6000\ln(Q+50)-60Q]_0^{50}=6000\ln(100)-3000-6000\ln 50=6000\ln\frac{100}{50}-3000=6000\ln 2-3000$

$PS=\int_0^{Q^*}\left(P^*-g(Q)\right)dQ=\int_0^{50}\left(60-(Q+10)\right)dQ=\int_0^{50}(50-Q)dQ=\left[50Q-\frac{1}{2}Q^2\right]_0^{50}=2500-1250=1250$

Extra Draw attention to the subtle use of Q and Q^* in each formula (and similar for P).