Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Tutorial 6 – Full solutions

Instruction

In a tutorial session of 2 hours, we will obviously not be able to discuss all questions. Therefore, the following procedure applies:

- we expect students to prepare all exercises in advance;
- we will dicuss only a selection of exercises;
- exercises that were not discussed during class are nevertheless part of the course;
- students can indicate their wish list of exercises to be discussed during the session;
- teachers may invite students to answer questions, orally or on the blackboard.

!!! We further understand that your time is limited, and in particular that your time between lecture and tutorial may be limited. In case you have no time to prepare everything, we kindly advise you to give priority to the exercises that are indicated with the !!! sign.

Multiple constrained optimization

!!! Q1 (Sydsæter & Hammond, 4/E, 14.6.7)

Solve the problem: max (min) $x+y$ subject to $\begin{cases} x^2+2y^2+z^2=1 \\ x+y+z=1 \end{cases}$

A1 $(0,0,1,\lambda=\cdots,\mu=\cdots)$ is a minimum and $\left(\frac{4}{5},\frac{2}{5},-\frac{1}{5},\lambda=\cdots,\mu=\cdots\right)$ is a maximum.

Sol The Lagrangian is $\mathcal{L}(x,y,z,\lambda,\mu)=x+y-\lambda(x^2+2y^2+z^2-1)-\mu(x+y+z-1)\Rightarrow$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial x}=1-2\lambda x-\mu=0 \\ \frac{\partial\mathcal{L}}{\partial y}=1-4\lambda y-\mu=0 \\ \frac{\partial\mathcal{L}}{\partial z}=-2\lambda z-\mu=0 \\ \frac{\partial\mathcal{L}}{\partial\lambda}=-x^2-2y^2-z^2+1=0 \\ \frac{\partial\mathcal{L}}{\partial\mu}=-x-y-z+1=0 \end{cases} \Rightarrow \begin{cases} 1-2\lambda x=\mu \\ 1-4\lambda y=\mu \\ -2\lambda z=\mu \\ x^2+2y^2+z^2-1=0 \\ x+y+z-1=0 \end{cases} \Rightarrow$$

$1-2\lambda x=1-4\lambda y\Rightarrow x=2y$

Substitute in third and fourth equation:

$$\begin{cases} 4y^2+2y^2+z^2-1=0 \\ 2y+y+z-1=0 \end{cases} \Rightarrow \begin{cases} 6y^2+z^2-1=0 \\ 3y+z-1=0 \end{cases} \Rightarrow 6y^2+1-6y+9y^2-1=0\Rightarrow$$

$3y(5y-2)=0\Rightarrow y=0\lor y=\frac{2}{5}\Rightarrow z=1\lor z=-\frac{1}{5}\Rightarrow x=0\lor x=\frac{4}{5}$

There are two stationary points of the Lagrangian: $(0,0,1,\lambda=\cdots,\mu=\cdots)$ and $\left(\frac{4}{5},\frac{2}{5},-\frac{1}{5},\lambda=\cdots,\mu=\cdots\right)$.

To investigate their nature, oberve that the objective function $x+y$ yields 0 in the former point, and $\frac{6}{5}$ in the latter point, so $(0,0,1)$ is a minimum and $\left(\frac{4}{5},\frac{2}{5},-\frac{1}{5}\right)$ is a maximum.

!!! Q2 (Sydsæter & Hammond, 4/E, 14.2.1)

Verify that $\frac{df^*(m)}{dm}=\lambda(m)$ holds for the problem max x^3y subject to $2x+3y=m$.

Sol The Lagrangian is $\mathcal{L}(x,y,\lambda)=x^3y-\lambda(2x+3y-m)\Rightarrow$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 3x^2y - 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = x^3 - 3\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -2x - 3y + m = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{3}{2}x^2y \\ \lambda = \frac{1}{3}x^3 \\ 2x + 3y = m \end{cases} \Rightarrow$$

$\frac{3}{2}x^2y = \frac{1}{3}x^3 \Rightarrow x^2 = 0 \vee \frac{3}{2}y = \frac{1}{3}x \Rightarrow x = 0 \vee y = \frac{2}{9}x$

Substitution of $x = 0$ gives $3y = m \Rightarrow y = \frac{m}{3}$; this gives the point $\left(0, \frac{m}{3}, 0\right)$

Substitution of $y = \frac{2}{9}x$ gives $\frac{8}{3}x = m \Rightarrow x = \frac{3}{8}m$; this gives the point $\left(\frac{3}{8}m, \frac{1}{12}m, \frac{9}{512}m^3\right)$

These two points are the only possible solutions to the max/min problems, respectively. The one which gives the largest value to the objective function f is: $\left(\frac{3}{8}m, \frac{1}{12}m\right)$

(because $f\left(0, \frac{m}{3}\right) = 0$ and $f\left(\frac{3}{8}m, \frac{1}{12}m\right) > 0$ (see below)).

The optimal value function: $f^*(m) = f\big(x^*(m), y^*(m)\big) = f\left(\frac{3}{8}m, \frac{1}{12}m\right) = \frac{9}{2048}m^4 \Rightarrow \frac{df^*(m)}{dm} = \frac{d}{dm}\left(\frac{9}{2048}m^4\right) = \frac{9}{512}m^3 = \lambda$. QED

Q3 (Sydsæter & Hammond, 4/E, 14.3.3)

(a) Find the solutions to the necessary conditions for the problem max(min) $f(x, y) = x + y$ subject to $g(x, y) = x^2 + y = 1$.

(b) Explain the solution geometrically by drawing appropriate level curves for $f(x, y)$ together with the graph of the parabola $x^2 + y = 1$. Does the associated minimization problem have a solution?

(c) Replace the constraint by $x^2 + y = 1.1$, and solve the problem in this case. Find the corresponding change in the optimal value of $f(x, y) = x + y$, and check to see if this change is approximately equal to $\lambda \times 0.1$, as suggested by the theory.

A3 (a) $\left(\frac{1}{2}, \frac{3}{4}\right)$ (b) there is no minimum (c) $\frac{1}{10}$

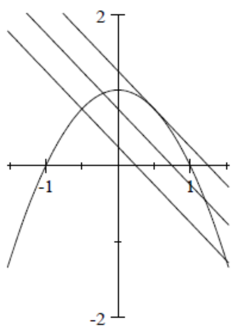
Sol

(a) The Lagrangian is $\mathcal{L}(x, y, \lambda) = x + y - \lambda(x^2 + y - 1) \Rightarrow$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 1 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -x^2 - y + 1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda x = \frac{1}{2} \\ \lambda = 1 \\ x^2 + y = 1 \end{cases} \Rightarrow$$

$x = \frac{1}{2} \Rightarrow y = 1 - \frac{1}{4} = \frac{3}{4}$; this gives the point $\left(\frac{1}{2}, \frac{3}{4}, 1\right)$

(b) The graph of the constraint $x^2 + y - 1 = 0$ or $y = 1 - x^2$ is a parabola with top (0,1)



The shown level curves (from right to the left) are $x+y=\frac{5}{4}$, $x+y=\frac{3}{4}$, and $x+y=\frac{1}{4}$. The level curve $\left(x+y=\frac{5}{4}\right)$ has only one point $\left(\frac{1}{2},\frac{3}{4}\right)$ in common with the constraint so the maximum of the objective function is at that point. (note: there is no minimum!)

(c) Now we find the point $\left(\frac{1}{2},\frac{17}{20}\right)$ for $\lambda=1$. The change in the value function is

$f^*(1.1)-f^*(1)=f\left(x^*(1.1),y^*(1.1)\right)-f\left(x^*(1),y^*(1)\right)=f\left(\frac{1}{2},\frac{17}{20}\right)-f\left(\frac{1}{2},\frac{3}{4}\right)=\left(\frac{1}{2}+\frac{17}{20}\right)-\left(\frac{1}{2}+\frac{3}{4}\right)=\frac{2}{20}=\frac{1}{10}=0.1\times\lambda$

Q4 (Sydsæter & Hammond, 4/E, 14.2.4)

(a) Solve the utility maximization problem $\max U(x,y)=\sqrt{x}+y$ subject to $x+4y=100$

(b) Suppose income increases from 100 to 101. What is the exact increase in the optimal value of $U(x,y)$? Compare with the value found in (a) for the Lagrange multiplier.

(c) Suppose we change the budget constraint to $px+qy=m$, but keep the same utility function. Derive the quantities demanded of the two goods if $m>q^2/4p$.

A4 (a) $(4,24)$ (b) $\frac{1}{4}$ (c) $\left(\frac{q^2}{4p^2},\frac{m}{q}-\frac{q}{4p}\right)$

Sol

(a) The Lagrangian is $\mathcal{L}(x,y,\lambda)=\sqrt{x}+y-\lambda(x+4y-100)\Rightarrow$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial x}=\frac{1}{2\sqrt{x}}-\lambda=0\\ \frac{\partial\mathcal{L}}{\partial y}=1-4\lambda=0\\ \frac{\partial\mathcal{L}}{\partial\lambda}=-x-4y+100=0\end{cases}\Rightarrow\begin{cases}\lambda=\frac{1}{2\sqrt{x}}\\ \lambda=\frac{1}{4}\\ x+4y=100\end{cases}\Rightarrow$$

$\frac{1}{2\sqrt{x}}=\frac{1}{4}\Rightarrow 2\sqrt{x}=4\Rightarrow\sqrt{x}=2\Rightarrow x=4\Rightarrow 4y=96\Rightarrow y=24\Rightarrow$ point $\left(4,24,\frac{1}{4}\right)$

(b) We now find: $x=4\Rightarrow 4y=97\Rightarrow y=\frac{97}{4}\Rightarrow$ point $\left(4,\frac{97}{4},\frac{1}{4}\right)$

$\Delta U=\sqrt{4}+97-\left(\sqrt{4}+24\right)=\frac{97}{4}-\frac{96}{4}=\frac{1}{4}=\lambda$

(note: there is exact equality here because U is linear in one of the variables)

(c) The Lagrangian is: $\mathcal{L}(x,y,\lambda)=\sqrt{x}+y-\lambda(px+qy-m)\Rightarrow$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial x}=\frac{1}{2\sqrt{x}}-\lambda p=0\\ \frac{\partial\mathcal{L}}{\partial y}=1-\lambda q=0\\ \frac{\partial\mathcal{L}}{\partial\lambda}=-px-qy+m=0\end{cases}\Rightarrow\begin{cases}\lambda=\frac{1}{2p\sqrt{x}}\\ \lambda=\frac{1}{q}\\ px+qy=m\end{cases}\Rightarrow$$

$\frac{1}{2p\sqrt{x}}=\frac{1}{q}\Rightarrow 2p\sqrt{x}=q\Rightarrow\sqrt{x}=\frac{q}{2p}\Rightarrow x=\frac{q^2}{4p^2}\Rightarrow qy=m-\frac{q^2}{4p^2}\Rightarrow y=\frac{m}{q}-\frac{q}{4p}\Rightarrow$

point $\left(\frac{q^2}{4p^2},\frac{m}{q}-\frac{q}{4p},\frac{1}{q}\right)$

(note: $y=\frac{m}{q}-\frac{q}{4p}>0\Rightarrow\frac{m}{q}>\frac{q}{4p}\Rightarrow m>\frac{q^2}{4p}$)

!!! Q5 (based on Sydsæter & Hammond, 4/E, 13.2.3)

Two weeks ago, a question was posed : “Solve the utility-maximizing problem $\max U=xyz$ subject to $x+3y+4z=108$, by making U a function of y and z by eliminating the variable x. Assume: $x>0$, $y>0$, and $z>0$.” At that time, it was solved using substitution. How would you proceed solving this question with Lagrange methode?

A5 maximum at $(36,12,9)$.

Sol Define the Lagrangian $\mathcal{L}(x,y,z,\lambda)=xyz-\lambda(x+3y+4z-108)$.

Stationary points of the Lagrangian are found by

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = yz - \lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = xz - 3\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z} = xy - 4\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = -x - 3y - 4z + 108 = 0 \end{cases}$$

The first two equations yield $3yz = 3\lambda = xz \Rightarrow x = 3y \Rightarrow y = \frac{1}{3}x$ (because $z \neq 0$).

Likewise, equations (1) and (3) yield $4yz = 4\lambda = xy \Rightarrow x = 4z \Rightarrow z = \frac{1}{4}x$.

These two results, when inserted in the equation (4) yields $x + 3 \cdot \frac{1}{3}x + 4 \cdot \frac{1}{4}x = 108 \Rightarrow 3x = 108 \Rightarrow x = 36$.

This immediately gives $y = 12$ and $z = 9$.

Proving that this point is indeed a maximum goes beyond the course, for functions of three variables.

Extra Compare this with the solution from week 4. There we reduced the problem in three variables (x, y, z) into one in two variables (y, z) by substituting the constraint in the maximization problem. Here we use the Lagrange method, thereby expanding the problem to a problem in four variables (x, y, z, λ). That is of course less efficient than the method of week 4. However, the method of week 4 will not work in many cases, so that the Lagrange method must be used.

Linear programming

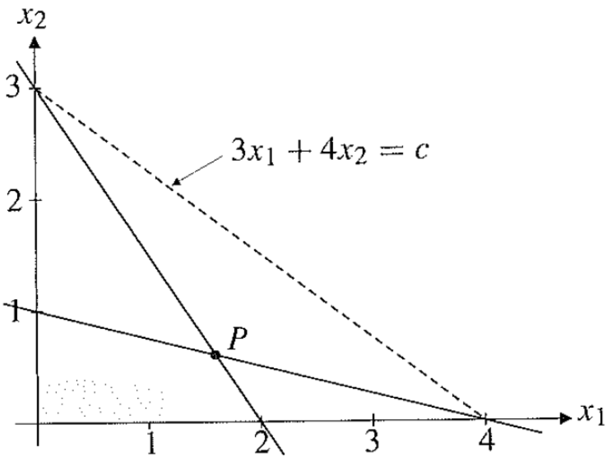
!!! Q1 (Sydsæter & Hammond, 4/E, 17.1.1.a)

Use the graphical method to solve the following LP problem:

$$\max 3x_1 + 4x_2 \text{ s.t. } \begin{cases} 3x_1 + 2x_2 \leq 6 \\ x_1 + 4x_2 \leq 4 \end{cases} \quad x_1 \geq 0, x_2 \geq 0$$

A1 $\left(\frac{8}{5}, \frac{3}{5}\right)$

Sol See the figure below.



The maximum occurs where the two lines intersect. This happens when $\begin{cases} 3x_1 + 2x_2 = 6 \\ x_1 + 4x_2 = 4 \end{cases} \Rightarrow$ $\begin{pmatrix} 3 & 2 & | & 6 \\ 1 & 4 & | & 4 \end{pmatrix} \xrightarrow{3r_2 - r_1} \begin{pmatrix} 3 & 2 & | & 6 \\ 0 & 10 & | & 6 \end{pmatrix} \Rightarrow x_2 = \frac{3}{5} \Rightarrow x_1 = \frac{8}{5}$. So the LP problem is solved in $\left(\frac{8}{5}, \frac{3}{5}\right)$.

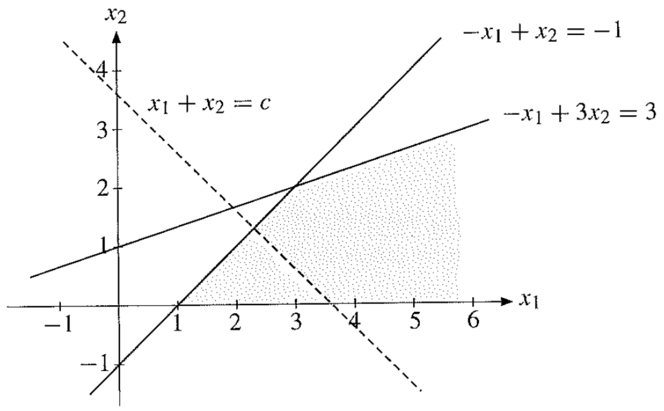
Q2 (Sydsæter & Hammond, 4/E, 17.1.4.a)

Is there a solution to the following problem?

$$\max x_1 + x_2 \text{ s.t. } \begin{cases} -x_1 + x_2 \leq -1 \\ -x_1 + 3x_2 \leq 3 \end{cases} \quad x_1 \geq 0, x_2 \geq 0$$

A2 No.

Sol See the figure below.



The feasible region is the shaded area. When the dashed line moves, c increases without limit while still having points in the shaded area.

Mathematical modeling & Miscellaneous topics

The best way to practice is by taking an old exam. We now put the exam of 22 October 2014 on the agenda. See BlackBoard for questions and solutions.