vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam: Business Mathematics / Quantitative Research Methods I

Code: E_BK1_BUSM / E_IBA1_BUSM / E_EBE1_QRM1

Examinator: dr. R. Heijungs
Co-reader: dr. G.J. Franx

Date: 21 October, 2015

Time: 12:00

Duration: 2 hours

Calculator allowed: Yes

Graphical calculator allowed: No

Number of questions: 3

Type of questions: Open / multiple choice

Answer in: Dutch or English (BK, EBE) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	25	23

Grades: 5 November, 2015.

Inspection: Will be announced on BlackBoard.

Number of pages: 7 (including front page and formula sheet)

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. **You must give an answer only**, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as $12, \frac{2}{3}$ and $\sqrt{3}e^{-2}$.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$1.2 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "mathematical formulation" means that you have to fill in a mathematical expression, such as "$0 \leq x \leq 1$" or "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "(B)".
- The indication "choose one or more" means that you have to choose one or more options, such as "(B) and (D) and (F)".

(a) Find the domain of the function $g(z) = \sqrt{\ln(z+1)}$. (mathematical formulation)

(b) Given is the function $f(p,q,r) = pq^r$. Find $\frac{\partial f}{\partial r}$. (mathematical formulation)

(c) The matrix equation $\mathbf{AX} - \mathbf{I} = \mathbf{B}'$ has a solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}' + \mathbf{A}^{-1}$. What conditions necessarily apply to $\mathbf{B}$? Choose one or more from "square", "invertible", "symmetric", "differentiable", and "none". (text)

(d) Evaluate the following integral: $\int_{-1}^{4}(2q+5)dq$. (exact)

(e) Given is a matrix $\mathbf{A}$ of order $(k \times l)$. The result of $\mathbf{AA}'$ is of order (4×4). What can you conclude on k and l? Answer something like "$k = l = 4$", "no conclusion for l and k", "$l > k$", etc. (mathematical formulation and/or text)

(f) Given is the function $f(j,n) = \prod_{i=j}^{n} i$, for $1 \leq j \leq n$ and $j,n \in \mathbb{N}^+$. Calculate $f(3,5)$. (exact)

(g) In the LP-case of the army's diet, the constraint $\sum_{i=1}^{m} a_{ji}x_i \geq c_j$ codes for the minimum intake of nutritional element j (proteins, vitamins, etc.) by eating an amount x_i of ingredient j (potatoes, meat, etc.). In this formula, a_{ji} denotes the amount of nutritional element j in ingredient i. For nutritional element 8 (salt), there is not only a minimum intake, but also a maximum intake, namely 25 units. Write the extra constraint inequality for the maximum salt intake. (mathematical formulation)

(h) Find all values of x (if any) for which $f(x) = e^{2x^2} - 1 = 0$. (exact or text)

(i) y is defined as a function of x by the following equation: $e^{xy} + x^2 - y^2 = 5$. Find a formula for $\frac{dy}{dx}$. (mathematical formulation)

(j) Simplify: $\ln \sqrt[3]{e^x}$. (mathematical formulation)

(k) Simplify as much as possible: $\sum_{i=1}^{n} 2x_i - \sum_{j=1}^{n} x_j$. (mathematical formulation or exact)

(l) Given is a matrix $\mathbf{A} = \begin{pmatrix} 2 & -5 & 45 \\ 1 & 4 & 1 \\ -1 & 20 & 13 \end{pmatrix}$ and $\mathbf{B} = (b_{i,j}) = \mathbf{A}^2$. Compute $b_{1,2}$. (exact)

(m) Given is a data vector $\mathbf{x}$ (with $s_x \neq 0$) and a data vector $\mathbf{y} = 3\mathbf{x}$. Determine, if possible, the value of $\frac{s_y^2}{s_x^2}$? (exact or text)

(n) Shoe size (s) is approximately related to body height (h) as $s = a + bh$. When s is in cm and h is in cm, what are the units of a and b? (text)

Bonus question: if you miss one of the above questions, you may still obtain maximum score by correctly answering the question below.

(o) What is the name of the following Greek letters: ω and ϕ? Write something like “alpha and beta”. (text)

Question 2 (25 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

Amusement parks, such as Disneyland and Walibi, are an important business type world-wide.

(a) Roller coasters are a critical asset: a spectacular roller coaster attracts more visitors, but fewer people will come when it gets too scary. An empirical formula is the following:

$$N = f(t, v) = t^3 - t^4 + 4tv - v^2 + 4v$$

where N (> 0) is the annual number of visitors, t (> 0) the track length (in kilometer), and v (> 0) the top speed (in meter/second). Determine the value of the top speed v that, with a given track length t, maximizes the number of visitors N, and find a function that expresses for every t the number of visitors N when that optimal top speed is chosen.

If you are unable to find the top speed, use $v = 3t + 1$.

[Read carefully: we do not ask you to find the values of t and v at which N assumes a maximum, but to find a function that gives the maximum N for every possible t!] (5 points)

(b) There are tickets for three age groups: infants (under 2 years), children (2-16 years old), and adults (16+). In addition, because visitors do not like long waiting lines, he or she can choose between a standard ticket and a VIP ticket. So, altogether there are six tariffs. Ticket prices are recorded in a vector ($\mathbf{p}$) of suitable order. Further, the sales of ticket type i on day number j is recorded as an element $n_{i,j}$ of a matrix $\mathbf{N}$, where j runs from 1 to 365. Give an expression for the total revenue in a year, either in vector/matrix form, or using the Σ notation. (5 points)

(c) The restaurant offers two meals: hamburgers (price in euro h) and French fries (price in euro f). Taking into account customer preferences, the restaurant's total revenue is given as

$$R = -200h^2 + 400hf - 10f^4 + 2$$

Is there a maximum revenue R? If so, for which values of h and f? (9 points)

(d) The swimming pool needs to be refilled. This is a time-consuming process. The water level L at time t is given by

$$L(t) = \int_0^t \phi(t')dt'$$

Here $\phi(t)$ is the filling rate at time t, given by $\phi(t) = 12e^{-2t}$. Find an expression for $L(t)$. (6 points)

Question 3 (23 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

Transport is a key sector in the Dutch economy.

(a) A bus company sells ticket to different destinations (Munich, Barcelona, etc.). The price p of a ticket is based on the travel distance d:

$$p = ad^b$$

where $a > 0$ and $b > 0$ are constants. The sales manager wants to charge a declining price per kilometer: a long trip is cheaper per kilometer than a short trip. Investigate what this implies for a and b. For instance, you may conclude that $a \leq b < 0$, or that $a > 0$ and that there is no restriction on b, etc. (5 points)

(b) A railway company can decide on how to allocate its vehicles to transporting passengers and transporting cargo. Profit π in millions of euro is given by

$$\pi = pc + 2p - 31$$

where p is millions of passengers, and c is tons of cargo. The availability of vehicles gives rise to the following budget constraint:

$$4p + 2c = 60$$

Use Lagrange's method to find the maximum profit. Include in your answer a proof that the point found is indeed a maximum.

[Throughout this question, you may treat the number of passengers p as a continuous variable.] (12 points)

(c) Econometricians have made an estimate of the demand function for air tickets:

$$q = 850 - 12p^{1.2}$$

where q is the quantity of air tickets sold and p a price level. Current equilibrium price is $p = 200$. Find the elasticity of the quantity sold with respect to price. (6 points)

Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE) Formula sheet (August 2015)

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d^2f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Function	**Derivative (with respect to x)**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x) \times g(x)$	$f'(x)g(x) + f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$	quotient rule
$f\left(g(x)\right)$	$f'\left(g(x)\right) \times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$CV_x = \frac{s_x}{\bar{x}} \quad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Functions and equations

$$a^x = e^{x \ln a}$$

$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \le \mathbf{b} \\ \text{and } \mathbf{x} \ge \mathbf{0} \end{cases}$$