Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Written exam – Full solutions 21 October 2015

Question 1

(a) $z \geq 0$ or $[0, \infty)$
because
$\ln(z + 1)$ should be ≥ 0 because the square root requires so. Thus, $z + 1 \geq 1$, so $z \geq 0$

(b) $pq^r \ln q$
because
$\frac{\partial}{\partial r}(pq^r) = p\frac{\partial}{\partial r}(q^r) = p\frac{\partial}{\partial r}\left(\left(e^{\ln q}\right)^r\right) = p\frac{\partial}{\partial r}\left(e^{r\ln q}\right) = pe^{r\ln q}\ln q = pq^r \ln q$

(c) square
because
B must have the same order as **I**, and by definition **I** is square. Thus **B** must be square. No further assumptions on **B** are needed.

(d) 40
because
$\int_{-1}^{4}(2q + 5)dq = \int_{-1}^{4} 2qdq + \int_{-1}^{4} 5dq = [q^2]_{-1}^{4} + [5q]_{-1}^{4} = 16 - 1 + 20 - -5 = 40$

(e) $k = 4$; no conclusion for l
because
If $\mathbf{A}$ is of order $(k \times l)$, so $\mathbf{A}'$ is of order $(l \times k)$ and $\mathbf{AA}'$ is of order $(k \times k)$. This is (4×4) if and only if $k = 4$. There is no condition on l.

(f) 60
because
$f(3,5) = \prod_{i=3}^{5} i = 3 \times 4 \times 5 = 60$

(g) $\sum_{i=1}^{m} a_{8i}x_i \leq 25$ or $\sum_{i=1}^{m} a_{8i}x_i < 25$
because
$\sum_{i=1}^{m} a_{8i}x_i$ is the total amount of salt intake, which should be no more than 25.

(h) 0
because
$e^{2x^2} - 1 = 0 \Rightarrow e^{2x^2} = 1 = e^0 \Rightarrow 2x^2 = 0 \Rightarrow x = 0$

(i) $-\frac{2x+ye^{xy}}{xe^{xy}-2y}$
because
$\frac{d}{dx}(e^{xy} + x^2 - y^2) = \frac{d}{dx}(5) \Rightarrow ye^{xy} + xe^{xy}\frac{dy}{dx} + 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x+ye^{xy}}{xe^{xy}-2y}$

(j) $\frac{1}{3}x$
because
$\ln \sqrt[3]{e^x} = \ln\left((e^x)^{1/3}\right) = \ln\left(e^{\frac{1}{3}x}\right) = \frac{1}{3}x$

(k) $\sum_{i=1}^{n} x_i$ or $\sum_{j=1}^{n} x_j$
because
$\sum_{i=1}^{n} 2x_i - \sum_{j=1}^{n} x_j = \sum_{i=1}^{n} 2x_i - \sum_{i=1}^{n} x_i = 2\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$

(l) 842
because
You only need to calculate $b_{1,2}$. It is given by $\sum_{i=1}^{3} a_{1i}a_{i2} = 2 \cdot -5 + -5 \cdot 4 + 45 \cdot 20 = -10 - 20 + 900 = 870$

(m) 9
because
$s_y^2 = (3s_x)^2 = 9s_x^2$, so $\frac{s_y^2}{s_x^2} = \frac{9s_x^2}{s_x^2} = 9$

(n) a is in cm and b is dimensionless (or: b has no unit)
because
s in cm implies a in cm and also bh in cm. Becasue h is in cm, b is dimensionless.

(o) omega and phi
because
that's just their name; check the Greek alphabet on BlackBoard

Question 2

(a) We first determine the top speed v at which N is a maximum, with given t:
$\frac{\partial f}{\partial v} = 4t - 2v + 4 = 0 \Rightarrow 2v = 4t + 4 \Rightarrow v = 2t + 2.$

With this value of v substituted in $f(t, v)$, we find
$N = f^*(t) = f(t, 2t + 2) = t^3 - t^4 + 4t(2t + 2) - (2t + 2)^2 + 4(2t + 2) =$
$t^3 - t^4 + 8t^2 + 8t - 4t^2 - 8t - 4 + 8t + 8 = t^3 - t^4 + 4t^2 + 8t + 4.$

In case you didn't find $v = 2t + 2$, use (as given)
$N = f^*(t) = f(t, 3t + 1) = t^3 - t^4 + 4t(3t + 1) - (3t + 1)^2 + 4(3t + 1) =$
$t^3 - t^4 + 12t^2 + 4t - 9t^2 - 6t - 1 + 12t + 4 = t^3 - t^4 + 3t^2 + 10t + 3$

(b) Revenue with tariff i on day j is given by $n_{i,j}p_i$. Summation over all tariff types yields $\sum_{i=1}^{6} n_{i,j}p_i$. Further aggregation over 365 days yields $\sum_{j=1}^{365}\sum_{i=1}^{6} n_{i,j}p_i$. This may also be written as $\sum_{j=1}^{365}\sum_{i=1}^{6} p_i n_{i,j}$.

Or in matrix notation as $\mathbf{p}'\mathbf{N}\mathbf{1}'$ or as $\mathbf{1}\mathbf{N}'\mathbf{p}$, where $\mathbf{1}$ is a vector of 365 numbers 1.

(c) $R = -200h^2 + 400hf - 10f^4 + 2$. To find a maximum, we try to find stationary values:
$\begin{cases} \frac{\partial R}{\partial h} = -400h + 400f = 0 \\ \frac{\partial R}{\partial f} = 400h - 40f^3 = 0 \end{cases} \Rightarrow \begin{cases} h = f \\ 10f = f^3 \end{cases} \Rightarrow f = h = 0 \vee \begin{cases} h = f \\ 10 = f^2 \end{cases} \Rightarrow$
$f = h = 0 \vee h = f = \pm\sqrt{10}.$

The negative and zero prices may be skipped, although you may also keep them (no restriction is stated in the text).

Next we must find out the nature of this stationary point. We use the second-order test.
$\frac{\partial^2 R}{\partial h^2} = -400, \frac{\partial^2 R}{\partial f^2} = -120f^2, \frac{\partial^2 R}{\partial h \partial f} = 400$, so $\frac{\partial^2 R}{\partial h^2}\frac{\partial^2 R}{\partial f^2} - \left(\frac{\partial^2 R}{\partial h \partial f}\right)^2 = 48000f^2 - 160000.$

For the solutions $f = h = \pm\sqrt{10}$, this is positive. Therefore, these stationary points are extreme values. Because $\frac{\partial^2 R}{\partial h^2} = -400 < 0$, it is a maximum.

The solution $f = h = 0$ gives $\frac{\partial^2 R}{\partial h^2}\frac{\partial^2 R}{\partial f^2} - \left(\frac{\partial^2 R}{\partial h \partial f}\right)^2 = -160000 < 0$. Therefore, this stationary point is a saddle point. However, as stated, you don't need to investigate this solution, because a zero price is not interesting.

(d) $L(t) = \int_0^t \phi(t')dt'$, and $\phi(t) = 12e^{-2t}$. So, we have $L(t) = \int_0^t 12e^{-2t'}dt'$.
This yields $L(t) = 12\int_0^t e^{-2t'}dt' = 12\left[-\frac{1}{2}e^{-2t'}\right]_0^t =$
$= 12\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}e^0\right) = 12\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) = 6(1 - e^{-2t}).$

Question 3

(a) Let $f(d)$ be the price per kilometer. Then $f(d) = \frac{p}{d} = \frac{ad^b}{d} = ad^{b-1}$.

If $f(d)$ is to be a declining function, $f'(d)$ must be negative. So $f'(d) = a(b-1)d^{b-2} < 0$.

The first term $a > 0$ (given in the text).
The third term $d^{b-2} > 0$ because a positive number d to some power is always positive.
So, the second term $b - 1 < 0$. This implies $b < 1$. It is further given that $b > 0$.
So the conclusion is that $0 < b < 1$, and there is no further restriction on a except $a > 0$.

(b) The problem is given by $\begin{cases} \max & \pi = pc + 2p - 31 \\ \text{s.t.} & 4p + 2c = 60 \end{cases}$.
Define the Lagrangian as $\mathcal{L}(p, c, \lambda) = pc + 2p - 31 - \lambda(4p + 2c - 60)$.

Stationary points of $\mathcal{L}$ occur when all three partial derivatives are zero. So:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = c + 2 - 4\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial c} = p - 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -4p - 2c + 60 = 0 \end{cases} \Rightarrow \begin{cases} \left.\begin{matrix} c + 2 = 4\lambda \\ p = 2\lambda \end{matrix}\right\} \Rightarrow c + 2 = 2p \\ 4p + 2c = 60 \end{cases} \Rightarrow 2c + 4 + 2c = 60$$

So $2c = 28 \Rightarrow c = 14$. And therefore $p = 8$, so $\pi = 97$.

The constraint is the equation of the line through $(p, c) = (0,30)$ and $(p, c) = (15,0)$.
At these points, $\pi = -31$ and $\pi = -1$ respectively. (Of course you can also choose two other points as long as $(8,14)$ is in between.)

Since the stationary point $(p, c) = (8,14)$ is between these two points, and $\pi(8,14) = 97$ is larger than -31 and -1, the stationary point must be a maximum.

(c) $q = f(p) = 850 - 12p^{1.2}$. The expression for elasticity is $\text{El}_p f(p) = \frac{p}{f(p)} \frac{df}{dp} = \frac{p}{f(p)} \cdot -14.4p^{0.2}$.
At $p = 200$, $f(p) = -6075$. Hence $\text{El}_p f(p) = \frac{200}{-6075} \cdot -41.5 = 1.37$.