vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam:	Business Mathematics / Quantitative Research Methods I
Code:	E_BK1_BUSM / E_IBA1_BUSM / E_EBE1_QRM1
Examinator:	dr. R. Heijungs
Co-reader:	dr. G.J. Franx
Date:	9 December, 2015
Time:	18:30
Duration:	2 hours
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	3
Type of questions:	Open
Answer in:	Dutch or English (BK, EBE) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	23	25

Grades:	24 December, 2015.
Inspection:	Will be announced on BlackBoard.
Number of pages:	7 (including front page and formula sheet)

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. **You must give an answer only**, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as 12, $\frac{2}{3}$ and $\sqrt{3}e^{-2}$.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$1.2 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "mathematical formulation" means that you have to fill in a mathematical expression, such as "$0 \leq x \leq 1$" or "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "(B)".
- The indication "choose one or more" means that you have to choose one or more options, such as "(B) and (D) and (F)".

(a) Given is a symmetric matrix $\mathbf{A}$ with $a_{2,3} = (\mathbf{A})_{2,3} = 5$ and a matrix $\mathbf{B} = \mathbf{I} - \mathbf{A}'$. What is $b_{2,3} = (\mathbf{B})_{2,3}$? The answer may be "not enough information". (exact or text)

(b) Given is a function $f(x) = 2x^2 + bx + c$. Further, it is known that $f(1) = f(3)$. Find b. (exact or text)

(c) Given is the function $f(x,y) = \frac{g(x,y)}{x} + y$, where $g(x,y)$ is another differentiable function. Express $\frac{\partial f}{\partial x}$ in terms of x, y, $g(x,y)$ and/or $\frac{\partial g}{\partial x}$. (mathematical formulation)

(d) Given is the matrix equation $\mathbf{A}^{-1}\mathbf{X} = \mathbf{I} + \mathbf{A} + \mathbf{B}'$, with $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ suitable matrices. Given an expression for $\mathbf{X}$. (mathematical formulation)

(e) The average house price in London is q GBP/ft^2. If 1 GPB is x EUR, and 1 ft = y metre, find the average house price in EUR/metre^2. (mathematical formulation)

(f) We can visualize the function $f(x,y) = x^2 + |y| + 3$ with level curves $f(x,y) = c$. For what values of c do level curves exist? (mathematical formulation)

(g) It is known that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. Find $\sum_{j=0}^{n-1} j$. (mathematical formulation)

(h) Solve for x: $\ln(x-3) = 1$. (exact)

(i) Given is a data vector $\mathbf{x}$ of length $n = 12$ with $\sum_i x_i = 60$ and $s_x^2 = 4$. Find CV_x. (exact)

(j) Determine $\int \left(\frac{z+1}{2}\right) dz$. (mathematical formulation)

(k) It is known that a function $f(x)$ has a maximum at $x = 5$ given by $f(5) = -12$. Give a first-order approximation for $f(5.1)$. The answer may be "not enough information". (exact or text or mathematical formulation)

(l) Given is a matrix $\mathbf{A} = (a)_{ij} = \begin{pmatrix} 3 & -2 & 7 \\ 0 & 4 & 1 \\ 5 & 8 & -2 \end{pmatrix}$. Find $\prod_{i=1}^{3} \prod_{j=1}^{3} a_{ij}$. (exact)

(m) For which values of x is the function $f(x) = -4x^3 + 12x + 5$ increasing? (mathematical formulation)

(n) Given is the function $f(p, q, r) = pq^r$. Find $\frac{\partial f}{\partial p}$. (mathematical formulation)

Bonus question: if you miss one of the above questions, you may still obtain maximum score by correctly answering the question below.

(o) Compute: $-3 \cdot 10^{-6}$ multiplied by 2 million multiplied by $2E0$. (exact)

Question 2 (23 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

The music industry is an important business worldwide.

(a) Concert halls (Concertgebouw, Paradiso, etc.) try to contract famous artists in order to raise ticket sales. But the fee that these celebrities ask may be too high for some potential visitors. Demand for tickets (Q) is given by

$$Q = 20S - \frac{5P}{S}$$

where P (> 0) is the price of a ticket and S (> 0) is the status of the artist on a YouTube-based popularity scale. Total revenue R is given by $R = PQ$.

For a given S, find the optimal value of P such that R is a maximum. Include in your answer a proof that the value is indeed a maximum. (5 points)

(b) Some shops broadcast loud music, with the hope that customers will buy more when there is music, and even more when the music is louder. For jeans shops, sales ($S > 0$ in euro/customer) depends on music volume ($L \geq 0$ in dB) as well as on the time ($T > 0$ in hours after opening) of the day (preference in the morning is different from that in the afternoon)

$$S = -11215 - 5T^2 - 3L^2 + 2TL - 60T + 376L$$

Find the value of the time after opening (T) and the music level (L) that together maximize sales. You do not need to consider possible maximum points at the boundary of the domain. (8 points)

(c) The assessor of a music school makes an inventory of the school's instruments. He arranges its assets in a vector $\mathbf{q}$. Each instrument is to be insured. Insurance price (in euro/yr) is organized in a vector $\mathbf{p}$. Altogether, $(\mathbf{p})_i = p_i$ is the number of instruments of type i, and $(\mathbf{q})_i = q_i$ is the insurance fee for one instrument of type i. The assessor of the school is unsure about the correct formula for total costs for insurance. For instance, it may be $\mathbf{pq}$ or $\mathbf{p}'\mathbf{q}$ or $(\mathbf{p}, \mathbf{q})$ or $(\mathbf{p}, \mathbf{q}')$, etc. Give two expressions in matrix form: one with a vector multiplication, one with an inner product. (4 points)

(d) Workers (crew and artists) at pop festivals may suffer from too loud music. Doctors have identified that cost ($C > 0$, in k€) related to hearing impairment (e.g., miscommunication, illness, stress) is given by

$$\frac{C^2}{L} - \frac{4C^2}{L^3} = 12$$

where $L > 0$ is the noise level (in dB). Determine a first-order approximation of $C(L)$ around $L = 80$. You may use the fact that $C = 31$ when $L = 80$. (6 points)

[Hint: find the relevant derivative at the point of interest.]

Question 3 (25 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

The Sinterklaas-time is an important period for Dutch businesses.

(a) The macro-economic impacts of Sinterklaas on Dutch businesses is modelled as follows:

$$\begin{cases} \Delta S = 3\Delta A + 4I + 70 \\ \Delta A = \frac{1}{3}\Delta S - 30 \\ I = \frac{1}{10}\Delta S - 7 \end{cases}$$

where ΔS is extra sales, ΔA is extra advertisement, and I is investments (for instance in hiring Sinterklaas). Write the model in matrix form, and solve the system. (8 points)

(b) There are two traditional kinds of treats: "pepernoten" and "marsepein". Let the price per kg of these be denoted by p and m respectively. A supermarket can set the prices such that profit is maximal. The profit function is

$$\pi = 200 - p^2 + pm - m$$

Market constraints, however, dictate that

$$pm = 16$$

Use Lagrange's method to find the optimal prices. Include in your analysis a proof that these prices indeed lead to a maximum profit. (12 points)

(c) Sales per day of gifts change over time: 10 days before December 5, sales take off, and they increase until December 5. The formula for sales per day, S, is

$$S = 60(t + 10)^2$$

with $t = -10$ corresponding to 10 days before December 5, and $t = 0$ corresponding to the last day, December 5. This expression is also valid for non-integer values of t, for instance for $t = -2.3$, etc. Find an expression for the total sales (accumulated over all days), using an integral. (5 points)

Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Formula sheet (August 2015)

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d^2f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Function	**Derivative (with respect to x)**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x)+g(x)$	$f'(x)+g'(x)$	sum rule
$f(x)\times g(x)$	$f'(x)g(x)+f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$	quotient rule
$f\left(g(x)\right)$	$f'\left(g(x)\right)\times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

$$CV_x = \frac{s_x}{\bar{x}} \quad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Functions and equations

$$a^x = e^{x \ln a}$$
$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$
$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$
$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$
$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \le \mathbf{b} \\ \text{and } \mathbf{x} \ge \mathbf{0} \end{cases}$$