Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Written exam – Full solutions 9 December 2015

Question 1

(a) -5
because
$\mathbf{A}$ is symmetric implies $\mathbf{A}' = \mathbf{A} \Rightarrow a_{i,j} = a_{j,i} \Rightarrow a_{2,3} = a_{3,2}$.
Further, $b_{2,3} = (\mathbf{I} - \mathbf{A}')_{2,3} = (\mathbf{I})_{2,3} - (\mathbf{A}')_{2,3} = 0 - (\mathbf{A})_{3,2} = -a_{3,2} = -a_{2,3} = -5$

(b) $b = -8$
because
$$\begin{cases} f(1) = 2 \cdot 1^2 + b \cdot 1 + c = 2 + b + c \\ f(3) = 2 \cdot 3^2 + b \cdot 3 + c = 18 + 3b + c \\ f(1) = f(3) \end{cases}$$
$\Rightarrow 2 + b + c = 18 + 3b + c \Rightarrow -16 = 2b \Rightarrow b = -8$

(c) $\frac{1}{x}\frac{\partial g}{\partial x} - \frac{g(x,y)}{x^2}$ or $\frac{x\frac{\partial g}{\partial x} - g(x,y)}{x^2}$
because
$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left(\frac{g(x,y)}{x} + y\right) = \frac{\partial}{\partial x}\left(\frac{g(x,y)}{x}\right) = \frac{1}{x}\frac{\partial g}{\partial x} - \frac{g(x,y)}{x^2}$

(d) $\mathbf{X} = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}\mathbf{B}'$ or $\mathbf{X} = \mathbf{A}(\mathbf{I} + \mathbf{A} + \mathbf{B}')$
because
$\mathbf{A}^{-1}\mathbf{X} = \mathbf{I} + \mathbf{A} + \mathbf{B}' \Rightarrow \mathbf{A}\mathbf{A}^{-1}\mathbf{X} = \mathbf{A}\mathbf{I} + \mathbf{A}\mathbf{A} + \mathbf{A}\mathbf{B}' \Rightarrow \mathbf{X} = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}\mathbf{B}'$

(e) $\frac{qx}{y^2}$
because
$q\frac{\text{GBP}}{\text{ft}^2}$ corresponds to $q\frac{x\text{ EUR}}{(y\text{ metre})^2}$, which is equal to $\frac{qx}{y^2}\frac{\text{EUR}}{\text{metre}^2}$

(f) $c \geq 3$ or $c \in [3, \infty)$ (or $c > 3$ or $c \in (3, \infty)$, because at $c = 3$ the curve is just a point)
because
$f(x,y) = x^2 + |y| + 3$. Level curves can only be made for possible values of f, so let's look at $x^2 + |y| + 3$. We have $x^2 \geq 0$, and $|y| \geq 0$, so $f(x,y) \geq 0 + 0 + 3 = 3$.

(g) $\frac{1}{2}(n-1)n$ or $\frac{1}{2}n^2 - \frac{1}{2}n$
because
$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, so $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$. Now, $\sum_{j=0}^{n-1} j = \sum_{j=1}^{n-1} j$, which is $\frac{1}{2}(n-1)n$
Alternatively, $\sum_{j=0}^{n-1} j = \sum_{j=1}^{n} j + (0 - n) = \frac{1}{2}n(n+1) + (0 - n) = \frac{1}{2}n^2 + \frac{1}{2}n - n = \frac{1}{2}n^2 - \frac{1}{2}n$

(h) $e + 3$
because
$\ln(x-3) = 1 \Rightarrow \ln(x-3) = \ln e \Rightarrow x - 3 = e \Rightarrow x = e + 3$

(i) $\frac{2}{5}$ or 0.4
because

$$CV_x = \frac{s_x}{\bar{x}} = \frac{\sqrt{s_x^2}}{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{\sqrt{4}}{\frac{1}{12}60} = \frac{2}{5}$$

(j) $\frac{1}{4}z^2 + \frac{1}{2}z + C$
because
$\int \left(\frac{z+1}{2}\right) dz = \int \frac{1}{2} z\, dz + \int \frac{1}{2} dz = \frac{1}{4}z^2 + C_1 + \frac{1}{2}z + C_2 = \frac{1}{4}z^2 + \frac{1}{2}z + C$
You can also work more directly: $\int \left(\frac{z+1}{2}\right) dz = \int \left(\frac{1}{2}z + \frac{1}{2}\right) dz = \frac{1}{4}z^2 + \frac{1}{2}z + C$

(k) -12
because
$f(5.1) \approx f(5) + f'(5)(5.1 - 5) = -12 + f'(5) \cdot 0.1$. Because $f(x)$ has a maximum at $x = 5$, we have $f'(5) = 0$. So $f(5.1) \approx -12$

(l) 0
because
$\prod_{i=1}^{3} \prod_{j=1}^{3} a_{ij} = (3 \times -2 \times 7) \times (0 \times 4 \times 1) \times (5 \times 8 \times -2) = 0$ (no need to calculate, because there is a 0 in the product)

(m) $-1 < x < 1$ or $x \in (-1,1)$ (or $-1 \leq x \leq 1$ or $x \in [-1,1]$)
because
$f(x) = -4x^3 + 12x + 5$ is increasing when $\frac{df}{dx} = -12x^2 + 12 > 0$. This yields $12x^2 < 12$, so $x^2 < 1$, so $-1 < x < 1$.
The book distinguishes increasing and strictly increasing functions; we also accept $-1 \leq x \leq 1$.

(n) q^r
because
$\frac{\partial f}{\partial p} = \frac{\partial pq^r}{\partial p} = q^r \frac{\partial p}{\partial p} = q^r$

(o) -12
because
2 million $= 2 \cdot 10^6$ and $2E0 = 2 \cdot 10^0 = 2$, so $-3 \cdot 10^{-6} \times 2 \cdot 10^6 \times 2 = -12 \cdot 10^0 = -12$

Question 2

(a) $R = PQ = P\left(20S - \frac{5P}{S}\right) = 20PS - \frac{5P^2}{S}$

With S given, R depends on P only: $R = f(P) = 20PS - \frac{5P^2}{S}$.

This function has a stationary point when $\frac{df}{dP} = 20S - \frac{10P}{S} = 0$.

This gives $\frac{10P}{S} = 20S$, so when $P = 2S^2$.

So $f(P) = 20PS - \frac{5P^2}{S}$ has a stationary value at $P = 2S^2$.

To study its nature, consider the second derivative, $\frac{d^2f}{dP^2} = -\frac{10}{S}$. Because $S > 0$ (see text), $\frac{d^2f}{dP^2} < 0$ everywhere, so also at $P = 2S^2$. Therefore, the stationary point $P = 2S^2$ is a maximum point.

Also possible: $\frac{df}{dp} > 0$ for $P < 2S^2$ and $\frac{df}{dp} < 0$ for $P > 2S^2$, therefore a maximum.

(b) $S = f(T, L) = -11215 - 5T^2 - 3L^2 + 2TL - 60T + 376L$

Stationary values require $\begin{cases} \frac{\partial f}{\partial T} = -10T + 2L - 60 = 0 \\ \frac{\partial f}{\partial L} = -6L + 2T + 376 = 0 \end{cases} \Leftrightarrow \begin{cases} 6L - 30T = 180 \\ -6L + 2T = -376 \end{cases} \Leftrightarrow 28T = 196$

So $T = 7$ and $L = 65$; this is the location of the only stationary point.

To analyze its nature, consider second-order derivatives: $\frac{\partial^2 f}{\partial T^2} = -10$, $\frac{\partial^2 f}{\partial L^2} = -6$, and $\frac{\partial^2 f}{\partial T \partial L} = 2$.

The second-order test yields $\frac{\partial^2 f}{\partial T^2} \cdot \frac{\partial^2 f}{\partial L^2} - \left(\frac{\partial^2 f}{\partial T \partial L}\right)^2 = (-10) \cdot (-6) - 2^2 = 60 - 4 = 56 > 0$.

So, the only stationary point is an extremum.

Finally, because $\frac{\partial^2 f}{\partial T^2} = -10 < 0$ (or because $\frac{\partial^2 f}{\partial L^2} = -6 < 0$), it is a maximum.

(c) There are q_i instruments of type i, and the insurance fee is p_i per instrument of type i. Insuring all instruments of type i therefore costs $q_i p_i$. Total insurance cost are $\sum_i q_i p_i$.

Using vector multiplication, this is $\mathbf{q}'\mathbf{p}$ or $\mathbf{p}'\mathbf{q}$.

Using the inner product, this is $(\mathbf{p}, \mathbf{q})$ or $(\mathbf{q}, \mathbf{p})$.

Note that the question is admittedly a bit confusing as it is written down: is $\mathbf{p}$ price or quantity, and is $\mathbf{q}$ quantity of price? In this case, it doesn't matter, because $\sum_i q_i p_i = \sum_i p q_i$.

(d) A first-order formula is $C(L) \approx C(80) + \frac{dC}{dL}(L - 80)$.

To calculate $\frac{dC}{dL}$, we use the implicit relation between C and L: $\frac{C^2}{L} - \frac{4C^2}{L^3} = 12$.

Differentiating with respect to L yields $\frac{d}{dL}\left(\frac{C^2}{L} - \frac{4C^2}{L^3}\right) = \frac{d}{dL}(12) \Leftrightarrow$

$\frac{2C}{L}\frac{dC}{dL} - \frac{C^2}{L^2} - \frac{8C}{L^3}\frac{dC}{dL} + \frac{12C^2}{L^4} = 0 \Leftrightarrow \left(\frac{2C}{L} - \frac{8C}{L^3}\right)\frac{dC}{dL} = \frac{C^2}{L^2} - \frac{12C^2}{L^4} \Leftrightarrow \frac{dC}{dL} = \frac{\frac{C^2}{L^2} - \frac{12C^2}{L^4}}{\frac{2C}{L} - \frac{8C}{L^3}} = \frac{C^2L^2 - 12C^2}{2CL^3 - 8CL} = \frac{C(L^2-12)}{L(2L^2-8)}$.

When $L = 80$ and $C = 31$, we find $\frac{dC}{dL} \approx 0.19$.

Therefore around $L = 80$, $C(L) \approx 31 + 0.19(L - 80) = 15.71 + 0.19L$.

Instead of using an implicit derivative, we may also try to go for an explicit equation of $C(L)$. To do so, rework $\frac{C^2}{L} - \frac{4C^2}{L^3} = 12$ into $C^2\left(\frac{1}{L} - \frac{4}{L^3}\right) = 12$. This yields $C^2 = \frac{12}{\left(\frac{1}{L} - \frac{4}{L^3}\right)}$ or $C = \pm\sqrt{\frac{12}{\left(\frac{1}{L} - \frac{4}{L^3}\right)}}$.

Only the positive solution must be considered (the negative solution is not in the allowed domain of C).
It is possible (of course) to differentiate this with respect to L, but it is not a very nice expression: $\frac{dC}{dL} = \frac{-\frac{1}{2}\sqrt{12}}{\left(\frac{1}{L}-\frac{4}{L^3}\right)^{3/2}}\left(\frac{-1}{L^2}+\frac{12}{L^4}\right)$ When $L = 80$ is filled out, the result is compatible with the previous one: $\frac{dC}{dL} \approx 0.19$.
Here is an example of a function that is more easily evaluated by using the implicit differentiation technique than by using straightforward differentiation.

Question 3

(a) The system of equations $\begin{cases} \Delta S = 3\Delta A + 4I + 70 \\ \Delta A = \frac{1}{3}\Delta S - 30 \\ I = \frac{1}{10}\Delta S - 7 \end{cases}$ can be written as

$$\begin{pmatrix} 1 & -3 & -4 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{10} & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta S \\ \Delta A \\ I \end{pmatrix} = \begin{pmatrix} 70 \\ -30 \\ -7 \end{pmatrix}$$

Using the Gaussian elimination procedure, we have

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & 70 \\ -\frac{1}{3} & 1 & 0 & -30 \\ -\frac{1}{10} & 0 & 1 & -7 \end{array}\right) \begin{array}{c} r_2 \times 3 \\ \sim \\ r_3 \times 10 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 70 \\ -1 & 3 & 0 & -90 \\ -1 & 0 & 10 & -70 \end{array}\right) \begin{array}{c} \sim \\ r_2 + r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & -4 & -20 \\ -1 & 3 & 0 & -90 \\ -1 & 0 & 10 & -70 \end{array}\right) \begin{array}{c} \sim \\ r_1 \times \frac{-1}{4} \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ -1 & 3 & 0 & -90 \\ -1 & 0 & 0 & -120 \end{array}\right) \begin{array}{c} \sim \\ r_3 - r_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ 0 & 3 & 0 & 30 \\ -1 & 0 & 0 & -120 \end{array}\right) \begin{array}{c} r_2 \times \frac{1}{3} \\ \sim \\ r_3 \times -1 \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 10 \\ 1 & 0 & 0 & 120 \end{array}\right) \begin{array}{c} \sim \\ r_1 \leftrightarrow r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

So $\Delta S = 120$, $\Delta A = 10$, and $I = 5$.

Of course, you may also solve the system by substitution:

$\Delta S = 3\left(\frac{1}{3}\Delta S - 30\right) + 4\left(\frac{1}{10}\Delta S - 7\right) + 70 = \Delta S - 90 + 0.4\Delta S - 28 + 70 = 1.4\Delta S - 48 \Rightarrow$
$0.4\Delta S = 48 \Rightarrow \Delta S = 120$.

Further, $\Delta A = \frac{1}{3} \cdot 120 - 30 = 10$ and $I = \frac{1}{10} \cdot 120 - 7 = 5$.

(b) The problem is given by $\begin{cases} \max & \pi = 200 - p^2 + pm - m \\ \text{s.t.} & pm = 16 \end{cases}$.

Define the Lagrangian as $\mathcal{L}(p, c, \lambda) = 200 - p^2 + pm - m - \lambda(pm - 16)$.

Stationary points of $\mathcal{L}$ occur when all three partial derivatives are zero. So:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial p} = -2p + m - \lambda m = 0 \\ \frac{\partial \mathcal{L}}{\partial m} = p - 1 - \lambda p = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -pm + 16 = 0 \end{cases} \Rightarrow \begin{cases} \lambda pm = -2p^2 + pm \\ \lambda pm = pm - m \\ pm = 16 \end{cases}$$

$$\Rightarrow \begin{cases} -2p^2 + pm = pm - m \\ pm = 16 \end{cases} \Rightarrow \begin{cases} 2p^2 = m \\ pm = 16 \end{cases} \Rightarrow \begin{cases} p^3 = 8 \\ m = \frac{16}{p} \end{cases}$$

So the coordinates of the stationary point of $\mathcal{L}$ are $p = \sqrt[3]{8} = 2$ and $m = \frac{16}{2} = 8$ (the coordinate λ is not needed). The profit in that point is $\pi = 200 - 2^2 + 2 \cdot 8 - 8 = 204$.

To prove that these coordinates maximize the profit, take two points on both sides of the stationary point that satisfy the constraint, for instance $(p, m) = (1,16)$, with $\pi = 199$ and $(p, m) = (16,1)$, with $\pi = -41$. The profit in both points is lower than the profit in the stationary point, which must therefore be a maximum.

(c) Sales per day is $S = 60(t+10)^2$. Accumulated from day $t = -10$ to day $t = 0$, the total sales is $\int_{-10}^{0} S dt = \int_{-10}^{0} 60(t+10)^2 dt$.

This can be further worked out as $60 \cdot \frac{1}{3}(t+10)^3\Big]_{-10}^{0} = 20(10^3 - 0^3) = 20000$.