vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam:	Business Mathematics
Code:	E_BK1_BUSM / E_IBA1_BUSM
Examinator:	dr. R. Heijungs
Co-reader:	dr. G.J. Franx
Date:	24 March, 2016
Time:	12:00
Duration:	2 hours
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	3
Type of questions:	Open / multiple chocie
Answer in:	Dutch or English (BK) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	22	26

Grades: The grades will be made public on: 11 April, 2016.

Inspection: Will be announced on BlackBoard.

Number of pages: 7 (including front page and formula sheet)

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. **You must give an answer only**, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as 12, $\frac{2}{3}$ and $\sqrt{3}e^{-2}$.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$1.2 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "mathematical formulation" means that you have to fill in a mathematical expression, such as "$0 \leq x \leq 1$" or "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "(B)".
- The indication "choose one or more" means that you have to choose one or more options, such as "(B) and (D) and (F)".

(a) Solve y from the following equation: $e^{(y^2-1)} = 1$ (exact).

(b) Evaluate the following integral: $\int_2^t \left(5 + \frac{1}{x}\right) dx$, where $t \geq 2$. (mathematical formulation)

(c) Given is that $\sum_{i=1}^{n} x_i = 24$. For which of the following expressions is the answer known? (choose one or more)

A) $\sum_{j=1}^{n} x_j$ B) $\sum_{i=1}^{m} x_i$ C) $\sum_{i=1}^{n} y_i$ D) $\prod_{i=1}^{n} x_i$ E) none of these

(d) For which values of x is the function $f(x) = \frac{\sqrt{x}}{1-\ln x}$ defined? (mathematical formulation)

(e) A square invertible matrix $\mathbf{Q}$ satisfies $\mathbf{AQ} = \mathbf{Q}$, where $\mathbf{A}$ is another matrix. Which of the following statements is necessarily true? (choose one or more)

A) $\mathbf{A} = \mathbf{Q}$ B) $\mathbf{A} = \mathbf{0}$ C) $\mathbf{A} = \mathbf{I}$ D) $\mathbf{A} = \mathbf{Q}^{-1}$ E) $\mathbf{A} = \mathbf{Q}'$ F) none of these

(f) Of a data vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ we know that $\bar{x} = 12$. Now, we define $\mathbf{y} = (x_1, x_2, \ldots, x_n, x_{n+1})$ with $x_{n+1} = 0$. What can you say about $\bar{y}$? (choose one)

A) $\bar{y} = \bar{x}$ B) $\bar{y} > \bar{x}$ C) $\bar{y} < \bar{x}$ D) nothing (not enough information)

(g) Given is the function $g(x, y, z) = x + yz + \frac{x}{yz}$. Determine $\frac{\partial g}{\partial y}$. (mathematical formulation)

(h) Given is the following matrix equation: $\mathbf{AX} + \mathbf{B}' = \mathbf{BA}$, where all matrices are invertible. Solve for $\mathbf{X}$. (mathematical formulation)

(i) Solve the following equation: $x = \frac{2}{3-x}$. (mathematical formulation)

(j) The cost (C, in €) of flying an airplane carrying a load W of cargo (in kg) over a distance of D km is given by $C = A + BDW$, where A en B are constants. What is the unit of B? The answer may be "not enough information". (mathematical formulation or text)

(k) Give the function value of the only stationary point of the function $f(x) = xe^{2x}$. (exact)

(l) A regression line is given as $z = at + b$, where t indicates time (in weeks) since some start moment t_0. We decide to reset the start moment at t_1. The new regression line is $z = ct' + d$, where t' indicates time (in weeks) since the new start moment t_1. Express coefficient c in terms of a, b, t_0 and/or t_1. (mathematical formulation)

(m) A matrix $\mathbf{A}$ of order $(m \times n)$ is subject to the following operation: $\mathbf{AA}'$. The result is of order (3×3). What can you conclude on m and n? Answer like "$m = 3; n > 3$", "no conclusion for $m; n = 9$", etc. (mathematical formulation or text)

(n) An implicit function is defined by $xy^2 - 2 + 3x^3 = 4$. Find $\frac{dy}{dx}$. (mathematical formulation)

Bonus question: if you miss one of the above questions, you may still obtain maximum score by correctly answering the question below.

(o) Write on your paper which of the five symbols belong(s) to the Greek alphabet (text):

Question 2 (22 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

Because the number of students at VU has increased since the main building was built, new elevators are under construction.

(a) There are 6 elevators and there are 5 time slots (8:45-9:00, 10:45-11:00, etc.) in a day. The number of students taking elevator number i at time slot j is recorded as N_{ij}. Give an expression for the total number of students T_i that take a specific elevator i over the day, using the Σ-notation. (4 points)

(b) The energy use per time unit ($W = \frac{dE}{dt}$) of a certain elevator depends on t according to the following function:

$$W(t) = -80t(t - 15) \qquad (\text{with } t \in [0,15])$$

Give an expression for the total energy (E) over the full 15 minute period. (6 points)

(c) A calculation model for the use of elevators is based on the following facts:

(i) the building has now $L = 6$ elevators, but this can be expanded to a maximum of $L = 9$ elevators

(ii) there are $N > 0$ students that need to take the elevator

(iii) it takes $0.1\frac{N}{\mathrm{L}}$ minutes before the last student is at his destination

(iv) everyone must be at his destination in 15 minutes

(v) building extra elevators costs K euro per elevator

Formulate a mathematical programming model that captures these facts and that minimizes cost. (5 points)

(d) Taking the stairs is an effective fitness training. With some exercise, you feel better, but excessive exercise will exhaust you. When body weight is B and the time of exercise on the stairs is T, the feel-good index F is given by

$$F = 200 - (4T - B)^2 \ln(5 + B)$$

For a person with a given body weight B, find the value of T that maximizes F, and calculate the value of F. If you don't manage to find the optimum, use $T = \frac{1}{2}B$. (7 points)

Question 3 (26 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

Studying hard gives you higher grades, but studying too hard is not good.

(a) It turns out that your grade (G) depends on the variables S (hours of study; $1 < S < 12$) and R (hours of relaxing; $1 < R < 12$) during the last day. An empirical formula is

$$G = -\frac{1}{4}R^4 + 8RS - 2S^2 - 60 - 16R + 8S$$

Find the number of hours of study and relaxing to obtain the highest grade. (10 points)

(b) Unfortunately, the student has only 2 hours available in total, because he has a side job. Now, we wish to maximize

$$G = -\frac{1}{4}R^4 + 8RS - 2S^2 - 60 - 16R + 8S$$

under the constraint

$$R + S = 2$$

Find the stationary point(s) of the Lagrangian. In this question, ignore the previous domain restrictions: $R, S \in \mathbb{R}$. You are not required to check the nature of the stationary point(s). (10 points)

(c) The student organizes his time budget, activities and priorities in a model. He distinguishes four variables: study (S), relaxing (R), household (H), and work (W). The model is given as

$$\begin{cases} S + R + H + W = 24 \\ W = 6 \\ 0.5H = 4 - S + W \\ S = 0.3H - R \end{cases}$$

Write the model as a matrix equation $\mathbf{Ax} = \mathbf{b}$, and specify $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$. (6 points)

Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Formula sheet (August 2015)

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d^2f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Function	**Derivative (with respect to x)**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x)+g(x)$	$f'(x)+g'(x)$	sum rule
$f(x)\times g(x)$	$f'(x)g(x)+f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$	quotient rule
$f\big(g(x)\big)$	$f'\big(g(x)\big)\times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

$$CV_x = \frac{s_x}{\bar{x}} \qquad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Functions and equations

$$a^x = e^{x \ln a}$$

$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \leq \mathbf{b} \\ \text{and } \mathbf{x} \geq \mathbf{0} \end{cases}$$