Business Mathematics (BK/IBA) – Quantitative Research Methods I (EBE)
Written exam – Full solutions 24 March 2016

Question 1

(a) ± 1
because
$e^{(y^2-1)} = 1 \Leftrightarrow e^{(y^2-1)} = e^0 \Leftrightarrow y^2 - 1 = 0 \Leftrightarrow y^2 = 1 \Leftrightarrow y = \pm 1$

(b) $5t - 10 + \ln t - \ln 2$
because
$\int_2^t \left(5 + \frac{1}{x}\right) dx = \int_2^t 5dx + \int_2^t \frac{1}{x} dx = 5x]_2^t + \ln x]_2^t = 5t - 10 + \ln t - \ln 2$

(c) A
because
i and j are just indices and can be substituted

(d) $x > 0 \wedge x \neq e$ or $x \in (0, e) \cup (e, \infty)$
because
$f(x) = \frac{\sqrt{x}}{1-\ln x}$ is defined when a) $x \geq 0$ because of $\sqrt{x}$, b) $x > 0$ because of $\ln x$, c) $1 - \ln x \neq 0$, so $\ln x \neq 1$, so $x \neq e$ because of the denominator. Putting this together gives $x > 0$ and $x \neq e$

(e) C
because
the identity matrix **I** is defined as the matrix that does nothing when multiplied with another matrix

(f) C
because
The average value will decrease if one element is added with a value that is smaller than the old average

(g) $z - \frac{x}{y^2 z}$
because
$\frac{\partial}{\partial y}\left(x + yz + \frac{x}{yz}\right) = \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}\left(\frac{x}{yz}\right) = 0 + z - \frac{x}{y^2 z} = z - \frac{x}{y^2 z}$

(h) $\mathbf{A}^{-1}\mathbf{B}\mathbf{A} - \mathbf{A}^{-1}\mathbf{B}'$ or $\mathbf{A}^{-1}(\mathbf{B}\mathbf{A} - \mathbf{B}')$
because
$\mathbf{A}\mathbf{X} + \mathbf{B}' = \mathbf{B}\mathbf{A} \Leftrightarrow \mathbf{A}\mathbf{X} = \mathbf{B}\mathbf{A} - \mathbf{B}' \Leftrightarrow \mathbf{A}^{-1}\mathbf{A}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A} - \mathbf{A}^{-1}\mathbf{B}' \Leftrightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A} - \mathbf{A}^{-1}\mathbf{B}'$

(i) 1 and 2
because
$x = \frac{2}{3-x} \Leftrightarrow (3-x)x = 2 \Leftrightarrow x^2 - 3x + 2 = 0 \Leftrightarrow (x-1)(x-2) = 0 \Leftrightarrow x = 1 \vee x = 2$. The expression is undefined for $x = 3$, but that is not of concern here.

(j) $\frac{\text{euro}}{\text{km} \times \text{kg}}$ or $\text{euro} \times \text{km}^{-1} \times \text{kg}^{-1}$

because
$C = A + BDW$. C in euro, so BDW in euro. DW in km × kg. So, B is in $\frac{\text{euro}}{\text{km}\times\text{kg}}$.

(k) $-\frac{1}{2e}$
because
$f(x) = xe^{2x} \Rightarrow f'(x) = e^{2x} + 2xe^{2x} = (1+2x)e^{2x}$. f has a stationary point when $(1 + 2x)e^{2x} = 0$, so when $x = -\frac{1}{2}$. The value of the function in that point is $-\frac{1}{2}e^{-1} = -\frac{1}{2e}$.

(l) $c = a$
because
Draw a straight line with slope a, and shift the vertical axis $t_1 - t_0$. Nothing will happen with the slope.
Alternatively, $z = at + b = ct' + d$ and $t' + t_1 = t + t_0$. So, $at + b = c(t + t_0 - t_1) + d$ for all t. Therefore, $at = ct$ and $b = c(t_0 - t_1) + d$, so $c = a$ and $d = b + a(t_1 - t_0)$.

(m) $m = 3$, no information on n (or: $n \in \mathbb{N}$)
because
$\mathbf{A}$ is of order $m \times n$, so $\mathbf{A}'$ is of order $n \times m$. The product $\mathbf{AA}'$ is of order $m \times m$. Therefore, $m = 3$. We can't say anything about n (except that it is an integer larger than 0).

(n) $\frac{y^2-9x^2}{2xy}$
because
$xy^2 - 2 + 3x^3 = 4$. Differentiating with respect to x yields $\frac{d}{dx}(xy^2 - 2 + 3x^3) = \frac{d}{dx}(4)$. Therefore, $y^2 + 2xy\frac{dy}{dx} + 9x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2-9x^2}{2xy}$.

(o) Ψ and ξ (ψ and ξ)
because
These are psi and xi. (The other characters are from the Cyrillic, Arabic, and Hindu alphabets.)

Question 2

(a) We must sum N_{ij} over all time slots ($j = 1, \dots, 5$), but not over i:

$$T_i = \sum_{j=1}^{5} N_{ij}$$

(b) The total energy consumed in the time interval is

$$E = \int_0^{15} W dt = \int_0^{15} -80t(t-15)dt$$

This can be worked out to

$$E = \int_0^{15} (-80t^2 + 1200t)dt = \int_0^{15} -80t^2 dt + \int_0^{15} 1200t dt = -\frac{80}{3}t^3\Big]_0^{15} + 600t^2]_0^{15}$$
$$= -90000 - 0 + 135000 - 0 = 45000$$

So $E = 45000$

(c) The objective function is cost $C = (L-6)K$ (because there are already 6 elevators). Further, the time required for the last student is $0.1\frac{N}{L}$. The decision variable is L.

So we formulate $\begin{cases} \min C = (L-6)K \\ \text{subject to } 6 \le L \le 9 \\ \text{and } 0.1\frac{N}{L} \le 15 \end{cases}$

(d) We have $F = 200 - (4T - B)^2 \ln(5 + B)$. When B is given, we optimize for T by setting $\frac{\partial F}{\partial T} = 0$. This gives $\frac{\partial F}{\partial T} = -2(4T - B)4\ln(5 + B) = 0 \Rightarrow 4T - B = 0 \Rightarrow T = \frac{1}{4}B$.

Another option would be $\ln(5 + B) = 0 \Rightarrow 5 + B = 1 \Rightarrow B = -4$, but this is not a feasible solution (negative body weight?), so we will ignore it.

We must check if this stationary point of F is a maximum. To do so, consider $\frac{\partial^2 F}{\partial T^2} = -32\ln(5 + B) < 0$, so the stationary point $(B, T) = \left(B, \frac{1}{4}B\right)$ is a maximum indeed.

With this value, we find $F = 200 - (4T - B)^2 \ln(5 + B) = 200$.

In case you couldn't find the value for T and used $T = \frac{1}{2}B$, you find $G = 200 - B^2 \ln(5 + B)$

Question 3

(a) $G = f(S,R) = -\frac{1}{4}R^4 + 8RS - 2S^2 - 60 - 16R + 8S$ has stationary points when

$$\begin{cases} \frac{\partial f}{\partial S} = 8R - 4S + 8 = 0 \\ \frac{\partial f}{\partial R} = -R^3 + 8S - 16 = 0 \end{cases} \Rightarrow \begin{cases} 16R - 8S + 16 = 0 \\ -R^3 + 8S - 16 = 0 \end{cases} \Rightarrow 16R - R^3 = 0 \Rightarrow R(16 - R^2) = 0 \Rightarrow$$

$R = 0 \vee R^2 = 16 \Rightarrow R = 0 \vee R = -4 \vee R = 4$. Only $R = 4$ is in the domain $1 < R < 24$.
The corresponding value of S is 10, which is in the domain $1 < S < 24$.

To find out the nature of these stationary points, consider the second order derivatives.

$$\frac{\partial^2 f}{\partial S^2}\frac{\partial^2 f}{\partial R^2} - \left(\frac{\partial^2 f}{\partial S \partial R}\right)^2 = -4 \times -3R^2 - (8)^2 = 12R^2 - 64.$$

$(S,R) = (10,4)$: $12R^2 - 64 = 1136 > 0$, so an extreme value; $\frac{\partial^2 f}{\partial S^2} = -4 < 0$ so a maximum.

So with $S = 10$ hours of study and $R = 4$ hours of relaxing, the student will receive a maximum grade.

(b) The problem is given by $\begin{cases} \max & G = -\frac{1}{4}R^4 + 8RS - 2S^2 - 60 - 16R + 8S \\ \text{s.t.} & R + S = 2 \end{cases}$.
Define the Lagrangian as $\mathcal{L}(S,R,\lambda) = -\frac{1}{4}R^4 + 8RS - 2S^2 - 60 - 16R + 8S - \lambda(R + S - 2)$.

Stationary points of $\mathcal{L}$ occur when all three partial derivatives are zero. So:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial S} = 8R - 4S + 8 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial R} = -R^3 + 8S - 16 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -R - S + 2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 8R - 4S + 8 \\ \lambda = -R^3 + 8S - 16 \\ R + S = 2 \end{cases}$$

$$\Rightarrow \begin{cases} 8R - 4S + 8 = -R^3 + 8S - 16 \\ R + S = 2 \end{cases} \Rightarrow \begin{cases} R^3 + 8R + 24 = 12S \\ S = 2 - R \end{cases} \Rightarrow R^3 + 8R + 24 = 12(2 - R)$$

The equation $R^3 + 20R = 0 \Rightarrow R(R^2 + 20) = 0$ has solution $R = 0$.
So the coordinates of the stationary point of $\mathcal{L}$ are $R = 0$ and $S = 2$ (the coordinate λ is not needed).

(c) The system $\begin{cases} S + R + H + W = 24 \\ W = 6 \\ 0.5H = 4 - S + W \\ S = 0.3H - R \end{cases}$ can be written relative to a basis vector $\mathbf{x} = \begin{pmatrix} S \\ R \\ H \\ W \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & -1 \\ 1 & 1 & -0.3 & 0 \end{pmatrix}\begin{pmatrix} S \\ R \\ H \\ W \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \\ 4 \\ 0 \end{pmatrix}.$$

So, $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & -1 \\ 1 & 1 & -0.3 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 24 \\ 6 \\ 4 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} S \\ R \\ H \\ W \end{pmatrix}$.