vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam: Business Mathematics / Quantitative Research Methods I

Code: E_BK1_BUSM / E_IBA1_BUSM / E_EBE1_QRM1

Examinator: dr. R. Heijungs
Co-reader: dr. G.J. Franx

Date: 22 October, 2014

Time: 12:00

Duration: 2 hours

Calculator allowed: Yes

Graphical calculator allowed: No

Number of questions: 3

Type of questions: Open / multiple choice

Answer in: Dutch or English (BK, EBE) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	28	20

Grades: 5 November, 2014.

Inspection: Will be announced on BlackBoard.

Number of pages: (7 (including front page and formula sheet))

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. You must give an answer only, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as 12, $\frac{2}{3}$, $\sqrt{3}$, and e^{-2}.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$12 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "formula" means that you have to fill in a mathematical expression, such as "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "B".
- The indication "choose one or more" means that you have to choose one or more options, such as "B and D and F".

(a) Solve x from: $6^{x-7} = \frac{1}{36}$. (exact)

(b) Given that $a_{ij} = 2 + 3i - 4j$, compute $\sum_{i=3}^{5} a_{ij}$. (exact or formula)

(c) Given is the expression $\int_s^t e^{\sqrt{x+z}} dz$. Which statement(s) is (are) correct? (choose one or more)

(A) The result is a function of s.

(B) The result is a function of t.

(C) The result is a function of x.

(D) The result is a function of z.

(E) None of the answers above is correct.

(d) Given is the matrix $\mathbf{R} = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 1 & 4 \\ 7 & -2 & 0 \end{pmatrix}$, and $\mathbf{A} = \mathbf{R}^2$. Calculate element $a_{3,1}$ from $\mathbf{A}$. (exact)

(e) You find data from different sources: $12 \cdot 10^4$, 5E5, 1.3 million, and 346,981.5. What is the sum of these four numbers? (2 significant digits)

(f) For a function $f(x)$, it is known that $\int f(x)dx = 2(x-1)^5 + \ln x + C$. Determine $f(x)$. (formula)

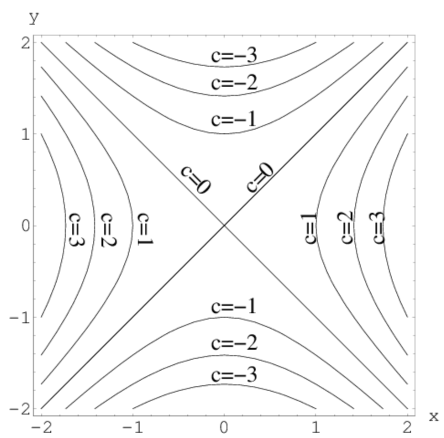
(g) It is known that $g(x)$ has a stationary point at the interior point $x = 3$, and that $g'(x) = x^2 - 2 + k$. What is k? (exact)

(h) Function q is given by $q(a,b) = a^2b - 3ab^2$. Determine $\frac{\partial^2 q}{\partial a \partial b}$. (formula)

(i) A data set $(\mathbf{x}, \mathbf{y})$, where the vector $\mathbf{x}$ denotes advertisement expenses (in euros) and the vector $\mathbf{y}$ denotes sales (in euros), is modelled by a regression equation $y = ax + b$. Which of the coefficients a and b (or none, or both) will change when we translate the currency from euros into Swedish crowns? (text)

(j) Apply Gauss-Jordan elimination to the augmented system $\begin{pmatrix} 1 & 0 & | & 3 \\ 3 & 5 & | & 5 \end{pmatrix}$ and write the results in the following way: $\begin{pmatrix} 1 & 0 & | & \cdots \\ 0 & 1 & | & \cdots \end{pmatrix}$. (formula)

(k) Given is a function $f(x,y)$ on $\mathbb{R} \times \mathbb{R}$. For a small part of its domain, it is visualized in a graph with the level curves below:



The derivative $\frac{dy}{dx}$ of $f(x,y) = -1$ in the point $(0,-1)$ is approximately: (choose one)

(A) $-\infty$.

(B) -1.

(C) 0.

(D) 1.

(E) ∞.

(l) Given is the matrix equation $\mathbf{X}'\mathbf{A} - \mathbf{B} = \mathbf{I}$, where $\mathbf{A}$ is an invertible matrix. Solve $\mathbf{X}$. (formula)

(m) Given is the curve $x^3y^2 + 4xy^3 = -8$. Find $\frac{dy}{dx}$. (formula)

(n) Given is that $\sum_{i=1}^{23} x_i = 34$, that $\sum_{i=23}^{45} x_i = 78$, that $x_{23} = 6$, and that $x_{24} = 2$.
Calculate $\sum_{i=1}^{45} x_i$. (exact)

Question 2 (28 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of page. You must specify all steps you take and use good notation principles.

(a) A manufacturer produces and sells two products in amounts x_1 and x_2. It has a profit function $\pi(x_1, x_2) = 4x_1x_2$ and a budget function $C(x_1, x_2) = x_1^2 + 4x_2^2$ (of which the level curves have the shape of an ellipse around the origin). Find the sale levels of x_1 and x_2, such that profit is maximized within a budget constraint of 32. Check the character of the stationary point (maximum or minimum) by using common sense (13 points)

(b) The company operates in a market where the demand function for product 1 is given by $x_1 = 40 - 6p_1 + 4\sqrt{p_2}.$, where p_1 is the price of product 1 and p_2 the price of product 2. Determine the elasticity of the demand of product 1 with respect to the price of product 2. Simplify the expression as far as possible. (5 points)

(c) Price data for products 1 and 2 are available, by observations on 251 days. The average prices are $\overline{p_1} = 23.6$ euro and $\overline{p_2} = 71.2$ euro, with variances $s_{p_1}^2 = 5.4$ euro^2 and $s_{p_2}^2 = 8.3$ euro^2. The correlation coefficient between the price vectors is $r_{p_1,p_2} = 0.71$. Compute the coefficient of variation for the price data of product 1 (CV_{p_1}) as well as the covariance between the two price vectors s_{p_1,p_2}. (6 points)

(d) The stock of products is indicated by a vector $\mathbf{s} = \binom{s_1}{s_2}$, where s_1 is the stock of product 1, and s_2 is the stock of product 2. The prices are as before: p_1 and p_2. Define a suitable price vector, and give a vector/matrix-formula for the total value of the stock. (4 points)

Question 3 (20 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of page. You must specify all steps you take and use good notation principles.

(a) A macro-economic model is defined by the equations

$$\begin{aligned} Y &= C + S + I \\ C &= C_0 + cY \\ S &= S_0 + sY \\ I &= iY \end{aligned}$$

where Y, C, S, and I are variables, and C_0, S_0, c, s, and i are constants. Write the model in matrix form $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{x} = \begin{pmatrix} Y \\ C \\ S \\ I \end{pmatrix}$ and $\mathbf{A}$ and $\mathbf{b}$ containing constants only. (5 points)

(b) If international trade is included in the model, two more variables must be introduced: E for export and M for import. Suppose that the first equation changes to $Y = C + S + I + E - M$. If no extra relationships are given, and we still want to write the system of equations as $\mathbf{Ax} = \mathbf{b}$, specify what the vector $\mathbf{x}$ will look like, and what happens to the order of $\mathbf{A}$ and $\mathbf{b}$. (5 points)

(c) A consultant expects that international debt might be correlated with national income. To investigate this, he collects international economic statistics in a data matrix $\mathbf{T}$. See a small excerpt of $\mathbf{T}$ (with first column and first row indicating the meaning of each row and column) below.

Country	**GDP/capita (USD/yr)**	**Inflation (%)**	**Debt/capita (USD)**	**Unemployment (%)**
Afghanistan	678	5.6	92	35
Belarus	...	...	...	...

Develop a formula with the summation symbol for the covariance between GDP/capita and debt/capita, using the appropriately indexed elements of the matrix $\mathbf{T}$. Introduce and define any symbols that you might need. (5 points)

(d) For a certain country, the debt/capita D in a certain year t is described by a function $f(t)$. The form of $f(t)$ is not known, but it is known that $f'(2014) = 3$ and $f''(2014) = -2$. Use a second-order approximation to find the debt/capita in 2016, if it is known than the debt/capita in 2014 is 350 USD. (5 points)

Business Mathematics/Quantitative Research Methods 1
Formula sheet (September 2014)

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$CV_x = \frac{s_x}{\bar{x}} \quad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\frac{d^2 f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

Function	**Derivative**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x) \times g(x)$	$f'(x)g(x) + f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$	quotient rule
$f(g(x))$	$f'(g(x)) \times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Equations

$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Functions

$$a^x = e^{x \ln a}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \leq \mathbf{b} \\ \text{and } \mathbf{x} \geq \mathbf{0} \end{cases}$$