vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam:	Business Mathematics / Quantitative Research Methods I
Code:	E_BK1_BUSM / E_IBA1_BUSM / E_EBE1_QRM1
Examinator:	dr. R. Heijungs
Co-reader:	dr. G.J. Franx
Date:	10 December, 2014
Time:	18:30
Duration:	2 hours
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	3
Type of questions:	Open / multiple choice
Answer in:	Dutch or English (BK, EBE) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	26	22

Grades:	7 January, 2015 (but we try earlier).
Inspection:	Will be announced on BlackBoard.
Number of pages:	(7 (including front page and formula sheet))

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. **You must give an answer only**, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as $12, \frac{2}{3}$ and $\sqrt{3}e^{-2}$.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$1.2 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "formula" means that you have to fill in a mathematical expression, such as "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "(B)".
- The indication "choose one or more" means that you have to choose one or more options, such as "(B) and (D) and (F)".

(a) If $f(q) = q + q^3$, calculate $\int_{-1}^{0} f(q)\, dq$. (exact)

(b) Given is the matrix equation $\mathbf{YX}' = \mathbf{YQY}$, where $\mathbf{Y}$ and $\mathbf{Q}$ are invertible square matrices. Give an expression for $\mathbf{X}$ that is simplified as much as possible. (formula)

(c) A function $f(x)$ that is defined on domain $x \in [a, b]$ is known to have a derivative $f'(x) > 0$ on that domain. Which statement(s) is (are) correct? (choose one or more)

(A) $f(x)$ has an interior maximum.

(B) $f(x)$ has a maximum at $x = a$

(C) $f(x)$ has a maximum at $x = b$.

(D) $f(x)$ has an interior minimum.

(E) $f(x)$ has a minimum at $x = a$.

(F) $f(x)$ has a minimum at $x = b$.

(G) None of the answers above is correct.

(d) A sample of $N = 321$ data points is collected, each data point p_i denotes the price of room rents per unit area in $\frac{€}{\mathrm{m}^2 \cdot \mathrm{month}}$ in university cities across the EU. What is the unit of the variance of this data vector? Write the result without parentheses, so without "(" and ")". (text)

(e) An LP-problem represents the minimization of costs (C) for a restaurant producing a pizza with salmon (index s), tomato (index t), and mozzarella (index m), with prices of the ingredients given by p_s, p_t and p_m (in €/kg ingredient) and the amount of each ingredient on the pizza given by a_s, a_t and a_m (in kg). The crust (i.e., the bottom of the pizza) costs 0.60 € for one pizza. State the objective function. (formula)

(f) A matrix $\mathbf{Q}$ of order 3×3 is known to be symmetric. Further, $q_{1,2} = 3$, $q_{1,3} = -2$, and $q_{2,3} = 0$, and $\sum_{i=1}^{3} q_{i,i} = 12$. Calculate $\sum_{i=1}^{3} \sum_{j=1}^{3} q_{i,j}$. (exact)

(g) Given is the equation of a curve in the (x, y)-plane $xy^3 + x^2y^2 = 2$. Calculate $\frac{dy}{dx}$ of this curve in point $(x, y) = (1,1)$. (exact)

(h) Matrices $\mathbf{A}$ and $\mathbf{B}$ both have order 4×3. What is the order of $\mathbf{A}'\mathbf{B}$? The answer may be "impossible". (formula or text)

(i) For a function $h(x, y)$ defined on $x > 0$ and $y > 0$, it is known that $\mathrm{El}_x h(x, y) = \frac{y}{\sqrt{x}}$, and that $h(x, y) > 0$ everywhere. Which statement(s) follow(s) logically from this? (choose one or more)

(A) $\frac{\partial h}{\partial x} = 0$ (D) $\frac{\partial h}{\partial y} = 0$

(B) $\frac{\partial h}{\partial x} > 0$ (E) $\frac{\partial h}{\partial y} > 0$ (G) None of the answers (A)-(F) is correct.

(C) $\frac{\partial h}{\partial x} < 0$ (F) $\frac{\partial h}{\partial y} < 0$

(j) Solve x from $\log x^5 = 20$. (exact)

(k) Apply Gauss-Jordan elimination to the augmented system $\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right)$ and write the results in the following way: $\left(\begin{array}{cc|c} 1 & 0 & \cdots \\ 0 & 1 & \cdots \end{array}\right)$. (formula)

(l) It is known that all solutions of a system of linear equations are given by $\left\{ \begin{pmatrix} \lambda \\ 1 - 2\lambda \\ 4\lambda - 2 \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$. Is the point $(-3, -5, 10)$ one of the solutions? (choose one)

(A) Yes (B) No (C) There is not enough information to answer this question.

(m) Find $\frac{\partial x}{\partial y}$ when $x(y, z) = 10^{zy}$. Simplify as far as possible. (formula)

(n) Unemployment at time t_0, written as $U(t_0)$, is big, and is still increasing, but less rapidly than before. How does this translate mathematically? (choose three)

(A) $U(t_0) > 0$ (E) $\left.\frac{dU}{dt}\right]_{t_0} > 0$ (I) $\left.\frac{d^2U}{dt^2}\right]_{t_0} > 0$

(B) $U(t_0) = 0$ (F) $\left.\frac{dU}{dt}\right]_{t_0} = 0$ (J) $\left.\frac{d^2U}{dt^2}\right]_{t_0} = 0$

(C) $U(t_0) < 0$ (G) $\left.\frac{dU}{dt}\right]_{t_0} < 0$ (K) $\left.\frac{d^2U}{dt^2}\right]_{t_0} < 0$

(D) no conclusion on $U(t_0)$ (H) no conclusion on $\left.\frac{dU}{dt}\right]_{t_0}$ (L) no conclusion on $\left.\frac{d^2U}{dt^2}\right]_{t_0}$

Question 2 (26 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

(a) The manager of a high-level restaurant investigates to what extent guests that order expensive wines also order expensive desserts. He collects data for 39 guests. He indicates the expenses on wine (in €) by a data vector $\mathbf{w}$ and on desserts (in €) by a data vector $\mathbf{d}$. For example, client 1 spent an amount w_1 on wine and an amount d_1 on desserts, etc. The manager finds that $\bar{w} = 23€$ and $\bar{d} = 13€$. Write down an expression for $s_{w,d}$, using the symbols given as much as possible, and filling in as much information as possible. Also specify the unit of the result. (5 points)

(b) The more a guest orders, the longer he occupies a table. Excluding wine and desserts, the formula for the time (in minutes) a guest stays in the restaurant is given by

$$T = 0.2\sqrt{a} + 0.6\ln m - 0.01(a + m)$$

where a is the expense on the appetizer and m the expense on the main course. ($a \geq 0$, $m > 0$, a and m in euro). Ignoring extreme values at the boundaries, find the extreme values of T (if any) and investigate their nature. (10 points)

(c) In a more realistic setting, the above formula for T is supplemented by a term $0.3d$ for the expense d on dessert ($d \geq 0$).The manager is composing a new daily menu. Because he wants to minimize the time spent in his restaurant by the customers, he is looking for those values of a, m, and d, that will minimize this time T. In doing so, he must keep in mind the culinary law of reasonable proportions, which states that the appetizer should not be more expensive than 60% of the price of the main course, and the dessert should never cost more than 50% of the main course. He also has to take into account the minimum prices of 7€ for appetizers, of 5€ for desserts, and of $14€$ for main courses. The maximum price of the entire menu is $55€$. Formulate a mathematical programming problem to this aim. (5 points)

(d) A customer card is introduced to increase turnover. After a customer's n^{th} visit, the discount at the next visit is $\sqrt[3]{10n}$ percent. Give a formula for the cumulative discount for a person who has had dinner m times, and who always orders meals and drinks which have a full (undiscounted) price of $60€$. (6 points)

Question 3 (22 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

(a) The market for Christmas trees (real trees with index r, artificial trees with index a) is described by the following equations:

$$\begin{aligned} Q_r &= 12 - 0.5p_r + 0.2p_a \\ Q_r &= 1.2p_r \\ Q_a &= 9 - 0.4p_a + 0.3p_r \\ Q_a &= 0.9p_a - 3 \end{aligned}$$

Here Q is the amount and p is the price for the two types of trees. Write this as a system of linear equations $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{x}' = (Q_r \quad Q_a \quad p_r \quad p_a)$, and $\mathbf{A}$ a matrix and $\mathbf{b}$ a vector consisting of constants only. (5 points)

(b) A more detailed model distinguishes two different sizes of Christmas trees: small (index s) and big (index b). This distinction is introduced both for real and for artificial trees. Thus, we have market information for four types of trees. If we want to use the same structure $\mathbf{Ax} = \mathbf{b}$, state the elements of $\boldsymbol{x}'$ and give the order of $\mathbf{A}$ and $\mathbf{b}$. (4 points)

(c) A grandmother invites her full family (children, grandchildren, etc.) for Christmas dinner. She serves wine: red (r) and white (w). The happiness she creates by serving these drinks is $H = 7R^{0.3}W^{0.7}$, where R is the number of liters of red wine, and W the number of liters of white wine. Both wines cost 5€ per liter. Because she has a small pension, her budget is limited to $5R + 5W = 250$€.

Lagrange's method yields an optimum mix of the two drinks: $R = 15$ and $W = 35$; the Lagrange multiplier is $\lambda = 0.75$, and the happiness $H = 190$ (you don't need to check these results).

Now, Grandma finds a 2 euro coin, so her budget is increased to 252€. Use a linear approximation in order to find the approximate new value of the happiness H_{new}. (8 points)

(d) The popularity of Christmas depends on climate: in Sweden it is much more popular than in Dubai. Spending (S, in dollar/capita) appears to be described by a function $f(T_F) = a + bT_F$, where T_F is the average December temperature (in Fahrenheit), and a and b are coefficients obtained by OLS regression.

To move from Fahrenheit to Celsius, subtract 32 and multiply the result by $\frac{5}{9}$. Restate the function $f(T_F)$ as a function $g(T_C)$, so as $g(T_C) = c + dT_C$, where T_C is the average December temperature at the Celsius scale. Also write down the units of the coefficients c and d. (5 points)

Business Mathematics/Quantitative Research Methods 1
Formula sheet (September 2014)

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$CV_x = \frac{s_x}{\bar{x}} \quad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$$

$$\frac{d^2 f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

Function	**Derivative**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x) \times g(x)$	$f'(x)g(x) + f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$	quotient rule
$f(g(x))$	$f'(g(x)) \times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Equations

$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Functions

$$a^x = e^{x \ln a}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)}f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \leq \mathbf{b} \\ \text{and } \mathbf{x} \geq \mathbf{0} \end{cases}$$