vrije Universiteit *amsterdam*

Faculty of Economics and Business Administration

Exam: Business Mathematics / Quantitative Research Methods I

Code: E_BK1_BUSM / E_IBA1_BUSM / E_EBE1_QRM1

Examinator: dr. R. Heijungs
Co-reader: dr. G.J. Franx

Date: 27 March, 2015

Time: 12:00

Duration: 2 hours

Calculator allowed: Yes

Graphical calculator allowed: No

Number of questions: 3

Type of questions: Open / multiple choice

Answer in: Dutch or English (BK, EBE) / English (IBA)

Remarks:
(1) You will receive a special answer sheet for question 1
(2) You will receive normal empty paper for questions 2 and 3
(3) Please write your name and student number (7 digits) on paper (1) and (2)
(4) You may keep the questions

Credit score:

Start	Question 1	Question 2	Question 3
10	42	25	23

Grades: 10 April, 2015.

Inspection: Will be announced on BlackBoard.

Number of pages: (7 (including front page and formula sheet))

Good luck!

Question 1 (42 points)

Question 1 consists of 14 short subquestions. Each subquestion counts for 3 points. **You must give an answer only**, on a separate special answer sheet. Note the following in answering the subquestions:

- The indication "exact" means that you have to fill in an exact number, such as $12, \frac{2}{3}$ and $\sqrt{3}e^{-2}$.
- The indication "1 decimal" means you have to fill in a number at the specified accuracy, such as "-23.0". In addition, you may have to specify additional text, such as "euro".
- The indication "2 significant digits" means you have to fill in a number at the specified accuracy, such as "$1.2 \cdot 10^3$". In addition, you may have to specify additional text, such as "euro".
- The indication "text", means you have to supply a phrase, such as "There is no stationary point".
- The indication "formula" means that you have to fill in a mathematical expression, such as "$\sqrt{a^2+1}$".
- The indication "choose one" means that you have to choose one option, such as "(B)".
- The indication "choose one or more" means that you have to choose one or more options, such as "(B) and (D) and (F)".

(a) Given are a matrix $\mathbf{A}$ of order 3×4 and a matrix $\mathbf{B}$ of order 4×4. What is the order of $\mathbf{A}^{-1}\mathbf{B}$? The answer may be "impossible". (formula or text)

(b) Given is the function $f(p,q) = e^{p^2+pq}$. Calculate $\frac{\partial f}{\partial p}$ at $(p,q) = (1,2)$. (exact)

(c) The system of linear equations $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} = \begin{pmatrix} -2 & 4 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ has a solution $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}$. What is the value of b_2? (exact)

(d) Find $\int_0^x (3-2y)dy$. (formula or exact)

(e) Determine both coordinates $(x, g(x))$ of the minimum point of the function $g(x) = e^{(2x-3)^2}$. (exact)

(f) It is known that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$. Which statements apply to $C = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$? (choose one or more)

(A) $C < 0$

(B) $C = 0$

(C) $0 < C < 1$

(D) $C = 1$

(E) $1 < C \leq 10$

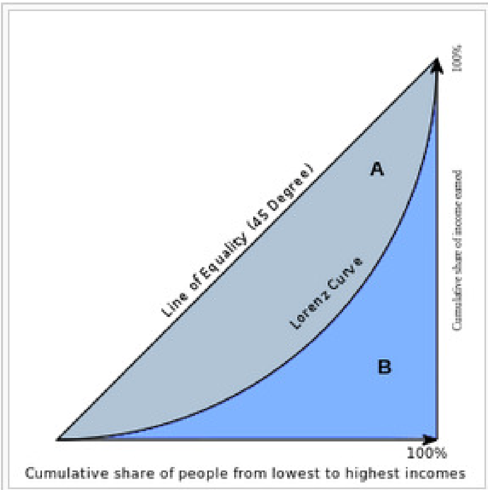
(F) $C > 10$

(G) There is not enough information available.

(g) Simplify: $\ln \frac{e^x e^y}{e^{2x}}$. (formula)

(h) Given is the function $S(k) = \sum_{i=k}^{2k}(k+i)$. Compute $S(3)$. (formula or exact)

(i) The Gini coefficient is an index of income inequality. Wikipedia illustrates it with the graph below. Let the line of equality be given by $y = x$ and the curved line by $y = f(x)$. Then the Gini coefficient equals the area A, given by ...... (choose one)



(A) $A = \int_0^{100\%}\big(xg(x)\big)dx$

(B) $A = \int_0^{100\%}\big(x + g(x)\big)dx$

(C) $A = \int_0^{x} g(x)dx$

(D) $A = \int_0^{100\%}\left(\frac{x}{g(x)}\right)dx$

(E) $A = \int_0^{100\%}\big(x - g(x)\big)dx$

(F) $A = \int_0^{y} g(x)dx$

(G) None of the above is correct.

(j) Someone claims that a data vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ has $\bar{x} = -23.6$ and $\sum_{i=1}^{n} x_i = 498.3$. Which statements can you deduce? (choose one or more)

(A) $n > 10$

(B) all data elements x_i are ≥ 0

(C) there must be a mistake in $\bar{x}$ and/or $\sum_{i=1}^{n} x_i$

(D) no data element x_i is > 500

(E) None of the above is correct.

(k) In a regression problem, a business analyst finds that $\ln y = 38.1 + 4.6\ \ln x$. He rewrites this equation as $y = ax^b$. Specify the value of coefficient b. (1 decimal)

(l) Find all three solutions for the equation in q: $q^3 - 2q^2 = 8q$. (exact)

(m) Calculate: $1500\ \text{million} + 2.5E8 + 3.5 \times 10^{10} + \frac{1}{10^{-9}}$. (3 significant digits)

(n) Determine the domain of the function $q(p) = \sqrt{1 - p^2}$. (formula or text)

Question 2 (25 points)

Question 2 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

(a) A fitness centre has different types of machines available (treadmill, home trainer, rowing machine, etc.), labelled 1 to 17. Each of these machines helps to train different physical aspects (leg muscles, arm muscles, heart, etc.), labelled 1 to 15. Data on the effectiveness is organized in a matrix $\mathbf{H}$, of order 17×15. An element h_{ij} indicates how much the use of 1 minute of machine i helps to improve physical aspect j. For a person, we define a vector $\mathbf{t}$ of which the elements contain the time that person uses each specific machine. We also define a vector $\mathbf{p}$ of which the elements tell how much a specific physical aspect is improved for that person. Give an expression in matrix notation that tells you how $\mathbf{p}$ is related to $\mathbf{t}$ and/or $\mathbf{H}$. (4 points)

(b') A busy salesman has exactly 30 minutes to do exercises. He dislikes most machines but only uses machines 1 and 2, during a time period t_1 and t_2. His doctor has told him that his overall fitness (f) is given by $f = (t_1)^{0.8}(t_2)^{0.4}$. Use Lagrange's method to find how the salesman should divide his 30 minutes of exercise over the two machines to obtain maximum health. (12 points)

(b) A busy salesman has exactly 30 minutes to do exercises. He is only interested in just two physical aspects: his heart and lungs. Further, he dislikes most machines but only uses machines 1 and 2. The improvement of his heart condition is given as $h = 12t_1 + 18t_2$, and of his lung condition as $l = 8t_1 + 6t_2$. His doctor has told him that his overall fitness (f) is given by $f = h^{0.8}l^{0.4}$. Determine how the salesman should divide his 30 minutes of exercise over the two machines to obtain maximum health. (12 points)

(c) The manager of the fitness centre has introduced a new marketing slogan for attracting new customers: "Your BMI is your discount". A person's BMI (body mass index) is defined as his body mass in kg, divided by the square of his height in m. The discount itself is calculated in euros. A visiting student from the US knows his height to be 71 inches and his body mass to be 165 pounds. He knows that 1 inch is 2.54 cm, 1 kg is 2.2 pound, and 1 euro is 1.3 dollar. What is his discount in dollars? (5 points)

(d) After having done heavy exercises, a group of friends take a final coke at the bar. The label of the bottle says that the beverage contains 128 calories. It is known how the weight (w in kg) of a person doing exercises relates to amount of calories burnt (c in calories) by that person and the duration of the exercises (t in minutes):

$$c = 0.25\, t\, w^{1.5}$$

Calculate the minimum time (in minutes) that a person of 64 kg should do exercises in order to lose at least the 128 calories from the final coke. (4 points)

Question 3 (23 points)

Question 3 must be answered on the empty exam sheets. Please start at the top of a page. You must **specify all steps** you take and **use good notation principles**.

(a) A bookshop owner sells newspapers for 2€. He buys them from the publisher for €1.20, and he can return unsold copies for €0.10. Indicating total profit in € by π, the number of sold copies by s and the number of returned copies by r, write an expression for $\pi(s, r)$. (4 points)

(b) The publisher proposes a new deal: newspapers can be purchased not for a fixed price, but for a per-copy price p that depends on the number of newspapers q purchased by the bookshop owner:

$$p = 1.30 \times e^{-kq}$$

where $k > 0$ is a constant set by the publisher. For simplicity, assume that p and q are continuous variables. Use calculus to show that for all feasible values of q, it is true that p decreases if q is increasing. (5 points)

(c) Besides the newspaper, the publisher publishes two magazines. The number of copies is very large, so you may again use continuous variable. The cost function is given by

$$C(A, B) = A^2 + 3B^2 - 3AB - 2A + 12$$

with A and B printing volumes of two different publications. Find the values of A and B for which the printing costs have a stationary point, and determine the nature this point. (9 points)

(d) In a more extended business case, a constrained optimization problem is introduced with 3 variables (A, B, C) and 2 constraints. The extreme values are found by introducing two Lagrange multipliers (λ and μ). The stationary points of the Lagrangian $\mathcal{L}(A, B, C, \lambda, \mu)$ are found by

$$\begin{cases} -3C - 8\lambda - 3 + 5B & = & 0 \\ 4A + 6B + 2\mu + 7 & = & 0 \\ 3\mu - 9\lambda + 1 & = & 0 \\ 4C - 2B & = & 0 \\ 2A - 7B & = & 0 \end{cases}$$

The analyst decides to solve this system of equations with matrix algebra, writing the system of equations in the form $\mathbf{Ax} = \mathbf{b}$. Specify what $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ look like. (5 points)

Business Mathematics/Quantitative Research Methods 1
Formula sheet (September 2014)

Descriptive statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$
$$s_{x,y} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$
$$CV_x = \frac{s_x}{\bar{x}} \quad r_{x,y} = \frac{s_{x,y}}{s_x s_y}$$

Derivatives

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$\frac{d^2 f(x)}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$
$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

Function	**Derivative**	**Remark**
A	0	constant function
$Af(x)$	$Af'(x)$	$A \in \mathbb{R}$
x^a	ax^{a-1}	$a \neq 0$
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x) \times g(x)$	$f'(x)g(x) + f(x)g'(x)$	product rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$	quotient rule
$f(g(x))$	$f'(g(x)) \times g'(x)$	chain rule
e^x	e^x	exponential function
$\ln\lvert x\rvert$	$\frac{1}{x}$	$x \neq 0$, logarithmic function

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} x_{i,j}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_{i,j}\right)$$

Equations

$$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integrals

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Matrices

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})' \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Functions

$$a^x = e^{x \ln a}$$

Approximations and elasticities

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

Extreme values

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

Constrained optimization

$$\begin{cases} \max f(\mathbf{x}) \\ \text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{c} \end{cases}$$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) \quad \frac{df^*}{dc} = \lambda$$

Curve fitting

$$y = ax + b$$

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad b = \frac{\sum y - a\sum x}{n}$$

Linear programming

$$\begin{cases} \max f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to } \mathbf{Ax} \le \mathbf{b} \\ \text{and } \mathbf{x} \ge \mathbf{0} \end{cases}$$