Business Mathematics (BK/IBA) – Quantitative Research Methods 1 (EBE)
Written exam – Full solutions 22 October 2014

Question 1

(a) 5
because
$6^{x-7} = \frac{1}{36} \Rightarrow 6^{x-7} = 6^{-2} \Rightarrow x - 7 = -2 \Rightarrow x = 5$

(b) $42 - 12j$
because
$\sum_{i=3}^{5}(2 + 3i - 4j) = \sum_{i=3}^{5} 2 + 3\sum_{i=3}^{5} i - 4\sum_{i=3}^{5} j =$
$= 3 \times 2 + 3 \times (3 + 4 + 5) - 3 \times 4j = 6 + 36 - 12j = 42 - 12j$

(c) (A) and (B) and (C)
because
If we change the value of *s, t,* or *x,* the outcome of the integral will also change. This is not the case with *z,* because the variable z will disappear due to integration. (In fact, the value of *z* cannot be chosen at all, since *z* is the integration variable that will run from *s* to *t*.)

(d) 12
because
$$\mathbf{R}^2 = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 1 & 4 \\ 7 & -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & -3 \\ 1 & 1 & 4 \\ 7 & -2 & 0 \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 7 \cdot 2 + -2 \cdot 1 + 0 \cdot 7 & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 12 & \cdots & \cdots \end{pmatrix}$$

(e) $23 \cdot 10^5$ or $2.3 \cdot 10^6$
because
$120{,}000 + 500{,}000 + 1{,}300{,}000 + 346{,}981.5 = 2{,}266{,}981.5$

(f) $10(x-1)^4 + \frac{1}{x}$
because
$\int f(x)dx = 2(x-1)^5 + \ln x + C \quad \Rightarrow \quad f(x) = \frac{d}{dx}\{2(x-1)^5 + \ln x + C\} = 10(x-1)^4 + \frac{1}{x}$

(g) -7
because
if $g(x)$ has a stationary point at an interior point at $x = 3$, it must be true that $g'(3) = 0$.
Therefore $3^2 - 2 + k = 0 \Rightarrow k = -7$.

(h) $2a - 6b$
because
$q(a,b) = a^2b - 3ab^2 \Rightarrow \frac{\partial^2 q}{\partial a \partial b} = \frac{\partial}{\partial a}\left(\frac{\partial q}{\partial b}\right) = \frac{\partial}{\partial a}(a^2 - 6ab) = 2a - 6b$

(i) b will change, but a will not change
because
when $\tilde{x} = kx$ and $\tilde{y} = ky$ (where k represents the conversion factor between euros and crowns), the regression equations becomes $\tilde{y} = ky = k(ax + b) = k\left(a\frac{\tilde{x}}{k} + b\right) = a\tilde{x} + kb$

(j) $\left(\begin{array}{cc|c}1 & 0 & 3 \\ 0 & 1 & -\frac{4}{5}\end{array}\right)$

because

$\left(\begin{array}{cc|c}1 & 0 & 3 \\ 3 & 5 & 5\end{array}\right) \xrightarrow{r_2-3r_1} \left(\begin{array}{cc|c}1 & 0 & 3 \\ 0 & 5 & -4\end{array}\right) \xrightarrow{r_2/5} \left(\begin{array}{cc|c}1 & 0 & 3 \\ 0 & 1 & -\frac{4}{5}\end{array}\right)$

(k) (C)

because

the level curves show $f(x,y) = c$ for different values of c, the value of $c = -1$ corresponds to a level curve that is horizontal in $(0,-1)$.

(l) $(\mathbf{A}^{-1})'(\mathbf{I}+\mathbf{B}')$ or $(\mathbf{A}')^{-1}(\mathbf{I}+\mathbf{B}')$

because

$\mathbf{X}'\mathbf{A}-\mathbf{B}=\mathbf{I} \Rightarrow \mathbf{X}'\mathbf{A}=\mathbf{I}+\mathbf{B} \Rightarrow \mathbf{X}'\mathbf{A}\mathbf{A}^{-1} = (\mathbf{I}+\mathbf{B})\mathbf{A}^{-1} \Rightarrow$

$\mathbf{X}' = (\mathbf{I}+\mathbf{B})\mathbf{A}^{-1} \Rightarrow \mathbf{X} = \left((\mathbf{I}+\mathbf{B})\mathbf{A}^{-1}\right)' = (\mathbf{A}^{-1})'(\mathbf{I}+\mathbf{B})' = (\mathbf{A}')^{-1}(\mathbf{I}+\mathbf{B}')$

(m) $-\frac{3x^2y^2+4y^3}{x^3 2y+12xy^2}$

because

$x^3y^2 + 4xy^3 = -8 \Rightarrow$

$\frac{d}{dx}(x^3y^2+4xy^3) = \frac{d}{dx}(-8) \Rightarrow 3x^2y^2 + x^3 2yy' + 4y^3 + 12xy^2y' = 0 \Rightarrow$

$(x^3 2y + 12xy^2)y' = -(3x^2y^2+4y^3) \Rightarrow y' = -\frac{3x^2y^2+4y^3}{x^3 2y+12xy^2}$

(n) 106

because

$\sum_{i=1}^{45} x_i = \sum_{i=1}^{23} x_i + \sum_{i=23}^{45} x_i - x_{23} = 34 + 78 - 6 = 106$

Question 2

(a) The problem is: $\begin{cases} \max \pi(x_1,x_2) = 4x_1x_2 \\ \text{subject to } x_1^2 + 4x_2^2 = 32 \end{cases}$.

The Lagrangian is $\mathcal{L}(x_1,x_2,\lambda) = 4x_1x_2 - \lambda(x_1^2+4x_2^2-32)$

and the stationary points require: $\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 4x_2 - 2\lambda x_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = 4x_1 - 8\lambda x_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -(x_1^2+4x_2^2-32) = 0 \end{cases}$

$\left.\begin{array}{l} 4x_2 = 2\lambda x_1 \Rightarrow \frac{2x_2}{x_1} = \lambda \\ 4x_1 = 8\lambda x_2 \Rightarrow \frac{x_1}{2x_2} = \lambda \end{array}\right\} \frac{2x_2}{x_1} = \frac{x_1}{2x_2} \Rightarrow 4x_2^2 = x_1^2 \Rightarrow (2x_2)^2 - x_1^2 = 0 \Rightarrow (2x_2+x_1)(2x_2-x_1) = 0 \Rightarrow$

$x_1 = \pm 2x_2$. Substituting this result into the constraint yields: $8x_2^2 = 32 \Rightarrow x_2^2 = 4 \Rightarrow x_2 = \pm 2$.

So, we find four stationary points with the following profit

$\pi(4,2) = 32$, $\pi(4,-2) = -32$, $\pi(-4,2) = -32$ and $\pi(-4,-2) = 32$.

Since the constraint is the equation of an ellipse around the origin, we conclude that the profit increases when circling the ellipse from $(4,-2)$ to $(4,2)$ and decreases from $(4,2)$ to $(-4,2)$. Since there are no stationary points in between, (4,2) must be a maximum. Since a manufacturer can't produce negative amounts of products, we conclude that the maximum (4,2) is the only acceptable solution.

(b) See formula sheet: $\text{El}_{p_2}x_1 = \frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2}$.

It is known that $x_1(p_1, p_2) = 40 - 6p_1 + 4\sqrt{p_2} = 40 - 6p_1 + 4p_2^{1/2}$, so $\frac{\partial x_1}{\partial p_2} = \frac{4}{2\sqrt{p_2}} = \frac{2}{\sqrt{p_2}}$.

Therefore, $\text{El}_{p_2}x_1 = \frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2} = \frac{p_2}{40-6p_1+4\sqrt{p_2}}\frac{2}{\sqrt{p_2}} = \frac{\sqrt{p_2}}{20-3p_1+2\sqrt{p_2}}$

(c) The coefficient of variation is defined by $CV_{p_1} = \frac{s_{p_1}}{\overline{p_1}} \Rightarrow CV_{p_1} = \frac{\sqrt{s_{p_1}^2}}{\overline{p_1}} = \frac{\sqrt{5.4}}{23.6} \approx 0.09847$

The covariance is related to the correlation coefficient by $r_{p_1,p_2} = \frac{s_{p_1,p_2}}{s_{p_1}s_{p_2}} \Rightarrow s_{p_1,p_2} = r_{p_1,p_2}s_{p_1}s_{p_2} = 0.71 \times \sqrt{5.4} \times \sqrt{8.3} \approx 4.7533$

(d) Define a price vector $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$. The total value ($V$) of the stock is $V = p_1 s_1 + p_2 s_2$, so $V = \mathbf{p} \cdot \mathbf{s}$, where $\cdot$ represents the inner product.

Question 3

(a) Rewrite the system as $\begin{cases} Y - C - S - I & = & 0 \\ -cY + C & = & C_0 \\ -sY \quad + S & = & S_0 \\ -iY \qquad + I & = & 0 \end{cases}$.

In matrix form, this can be formulated as $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -c & 1 & 0 & 0 \\ -s & 0 & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y \\ C \\ S \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ C_0 \\ S_0 \\ 0 \end{pmatrix}$

(b) $\mathbf{x}$ will change into $\begin{pmatrix} Y \\ C \\ S \\ I \\ E \\ M \end{pmatrix}$. The order of $\mathbf{A}$ will become 4×6, and the order of $\mathbf{b}$ will remain at 4.

(c) If n denotes the number of rows of the data matrix $\mathbf{T}$, the covariance between GDP/capita and debt capita will be found by

$$\frac{1}{n-1}\sum_{i=1}^{n}(t_{i,1} - \overline{t_1})(t_{i,3} - \overline{t_3})$$

In this formula we used $\overline{t_1} = \frac{1}{n}\sum_{i=1}^{n} t_{i,1}$, and $\overline{t_3} = \frac{1}{n}\sum_{i=1}^{n} t_{i,3}$.

(d) $f(2016) \approx f(2014) + f'(2014)(2016 - 2014) + \frac{1}{2}f''(2014)(2016 - 2014)^2 =$
$= 350 + 3 \times 2 + \frac{1}{2} \times (-2) \times 2^2 = 350 + 6 - 4 = 352$