Business Mathematics (BK/IBA) – Quantitative Research Methods 1 (EBE)
Written exam – Full solutions 10 December 2014

Question 1

(a) $-\frac{3}{4}$

$\int_{-1}^{0}(q+q^3)dq = \int_{-1}^{0} q dq + \int_{-1}^{0} q^3 dq = \left[\frac{1}{2}q^2 + \frac{1}{4}q^4\right]_{-1}^{0} = \frac{1}{2}(0-1) + \frac{1}{4}(0-1) = -\frac{3}{4}$

(b) $\mathbf{Y}'\mathbf{Q}'$

$\mathbf{YX}' = \mathbf{YQY} \Rightarrow \mathbf{Y}^{-1}\mathbf{YX}' = \mathbf{Y}^{-1}\mathbf{YQY} \Rightarrow \mathbf{X}' = \mathbf{QY} \Rightarrow \mathbf{X} = (\mathbf{QY})' = \mathbf{Y}'\mathbf{Q}'$

(c) (C) and (E)

If $f'(x) > 0$ everywhere on $[a,b]$, $f(x)$ is increasing between a and b. Hence, there is no interior point between a and b for which is $f(x)$ has an extreme value, and $f(a)$ is the minimum and $f(b)$ the maximum.

(d) $\frac{€^2}{\text{m}^4\cdot\text{month}^2}$

The variance is always in squared units (compared to the data itself). Here, the data is in $\frac{€}{\text{m}^2\cdot\text{month}}$, so its variance is in $\left(\frac{€}{\text{m}^2\cdot\text{month}}\right)^2 = \frac{€^2}{\text{m}^4\cdot\text{month}^2}$.

(e) $C(a_s, a_t, a_m) = 0.60 + a_s p_s + a_t p_t + a_m p_m$

The costs are given by $C = 0.60 + a_s p_s + a_t p_t + a_m p_m$. It is a function of the amount of the three ingredients, a_s, a_t, and a_m.

(f) 14

Write the matrix as $\mathbf{Q} = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix}$. The matrix is symmetric, so we have $q_{1,2} = q_{2,1} (= 3)$, $q_{1,3} = q_{3,1} (= -2)$, and $q_{2,3} = q_{3,2} (= 0)$. So $\mathbf{Q} = \begin{pmatrix} q_{1,1} & 3 & -2 \\ 3 & q_{2,2} & 0 \\ -2 & 0 & q_{3,3} \end{pmatrix}$.
The objective is to find $\sum_{i=1}^{3}\sum_{j=1}^{3} q_{i,j}$, which is the sum of all 9 elements. We easily see that it is $q_{1,1} + q_{2,2} + q_{33} + 2 \times (3 - 2 + 0) = q_{1,1} + q_{2,2} + q_{33} + 2$.
Finally, we know that $\sum_{i=1}^{3} q_{i,i} = q_{1,1} + q_{2,2} + q_{33} = 12$. So, the sum of all elements is $12 + 2 = 14$.

(g) $-\frac{3}{5}$

$xy^3 + x^2y^2 = 2 \Rightarrow \frac{d}{dx}(xy^3 + x^2y^2) = \frac{d}{dx}(2) \Rightarrow y^3 + 3xy^2\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} = 0 \Rightarrow$

$y^3 + 2xy^2 + (3xy^2 + 2x^2y)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^3+2xy^2}{3xy^2+2x^2y} \Rightarrow$ in point (1,1): $\left.\frac{dy}{dx}\right]_{(1,1)} = -\frac{1+2}{3+2} = -\frac{3}{5}$

(h) 3×3

$\mathbf{A}'$ is of order 3×4, $\mathbf{B}$ of order 4×3, so $\mathbf{A}'\mathbf{B}$ is of order 3×3.

(i) (B)

$El_x h(x,y) = \frac{x}{h(x,y)}\frac{\partial h}{\partial x} = \frac{y}{\sqrt{x}} \Rightarrow \frac{\partial h}{\partial x} = \frac{y}{\sqrt{x}}\frac{h(x,y)}{x}$. Now, it is clear that $\frac{y}{\sqrt{x}} > 0$ and that $\frac{h(x,y)}{x} > 0$. Hence their product $\frac{\partial h}{\partial x} = \frac{y}{\sqrt{x}}\frac{h(x,y)}{x} > 0$. We don't know anything about $\frac{\partial h}{\partial y}$.

(j) 10000

$\log x^5 = 20 \Rightarrow 5\log x = 20 \Rightarrow \log x = 4 \Rightarrow x = 10^4 = 10000$

(k) $\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -\frac{1}{2} \end{array}\right)$

$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right) \xrightarrow{r_2-4r_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -2 & -4 & 1 \end{array}\right) \xrightarrow{r_2/(-2)} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -\frac{1}{2} \end{array}\right)$

(l) (B)

$(\lambda, 1-2\lambda, 4\lambda-2) = (-3,-5,10)$ implies that $\begin{cases} \lambda = -3 \\ 1-2\lambda = -5 \\ 4\lambda - 2 = 10 \end{cases}$. However, $\lambda = -3$ inserted in the second equation gives $7 = -5$, which is a contradiction. Therefore, $(-3,-5,10)$ can't be a solution.

(m) $z\ln(10)\ 10^{zy}$

$x(y,z) = 10^{zy} = e^{zy\ln(10)} \Rightarrow \frac{\partial x}{\partial y} = z\ln(10)\ e^{zy\ln(10)} = z\ln(10)\ 10^{zy}$

(n) (A) and (E) and (K)

U is big, which we may interpret as larger than 0, it is increasing over time, so $\frac{dU}{dt}$ is positive, and the rate of increase is decreasing, so $\frac{d^2U}{dt^2}$ is negative.

Question 2

(a) $s_{w,d} = \frac{1}{n-1}\sum_{i=1}^{n}(w_i - \bar{w})(d_i - \bar{d}) = \frac{1}{39-1}\sum_{i=1}^{39}(w_i - 23)(d_i - 13) = \frac{1}{38}\sum_{i=1}^{39}(w_i - 23)(d_i - 13)$.
The unit of $s_{w,d}$ is €^{2}.

(b) $T(a,m) = 0.2\sqrt{a} + 0.6\ln m - 0.01(a+m) = 0.2\,a^{1/2} + 0.6\ln m - 0.01(a+m)$.

Stationary points require $\begin{cases}\frac{\partial T}{\partial a} = 0.1\,a^{-1/2} - 0.01 = 0 \\ \frac{\partial T}{\partial m} = \frac{0.6}{m} - 0.01 = 0\end{cases} \Rightarrow \begin{cases}\frac{0.1}{\sqrt{a}} = 0.01 \Rightarrow \sqrt{a} = \frac{0.1}{0.01} = 10 \Rightarrow a = 100 \\ \frac{0.6}{m} = 0.01 \Rightarrow m = \frac{0.6}{0.01} = 60\end{cases}$

The only stationary point of T is at $(a,m) = (100,60)$. Its value is $T(100,60) = 0.6\ln 60 + 0.4$.
To investigate the nature of this stationary point, consider $\frac{\partial^2 T}{\partial a^2}\frac{\partial^2 T}{\partial m^2} - \left(\frac{\partial^2 T}{\partial a \partial m}\right)^2\Big|_{(100,60)}$:
$\frac{\partial^2 T}{\partial a^2} = -0.05a^{-3/2}$; $\frac{\partial^2 T}{\partial m^2} = \frac{-0.6}{m^2}$; $\frac{\partial^2 T}{\partial a \partial m} = 0$.
Hence $\left\{\frac{\partial^2 T}{\partial a^2}\frac{\partial^2 T}{\partial m^2} - \left(\frac{\partial^2 T}{\partial a \partial m}\right)^2\right\}\Big|_{(100,60)} = 0.03\frac{a^{-3/2}}{m^2}\Big|_{(100,60)} = 0.03\frac{100^{-3/2}}{60^2} > 0$.
The point $(100,60)$ is therefore an extreme value.
To investigate if it is a maximum or a minimum, choose either $\frac{\partial^2 T}{\partial a^2}\Big|_{(100,60)} = -0.05 \times 10^{-\frac{3}{2}} < 0$ or $\frac{\partial^2 T}{\partial m^2}\Big|_{(100,60)} = \frac{-0.6}{60^2} < 0$. Both are negative.
So, the point $(100,60)$ is a maximum; it is the only maximum.

(c) The information given translates into
(1) $T = 0.2\sqrt{a} + 0.6\ln m - 0.01(a+m) + 0.3d$
(2) T is to be minimized
(3) $a \le 0.6m$
(4) $d \le 0.5m$
(5) $a \ge 7$
(6) $m \ge 14$
(7) $d \ge 5$
(8) $a + m + d \le 55$
So the mathematical programming problem becomes:

$$\text{minimize } 0.2\sqrt{a} + 0.6\ln m - 0.01(a+m) + 0.3d$$
$$\text{subject to} \begin{cases} a + m + d \le 55 \\ a \ge 7 \\ a \le 0.6m \\ m \ge 14 \\ d \ge 5 \\ d \le 0.5m \end{cases}$$

(d) The discount at the n^{th} dinner is $D_n = 0.01 \times \sqrt[3]{10n} \times 60 = 0.6 \times \sqrt[3]{10} \times \sqrt[3]{n}$.
So the cumulative discount for the first m dinners yields $\sum_{n=1}^{m-1} D_n = 0.6\sqrt[3]{10}\sum_{n=1}^{m-1}\sqrt[3]{n}$.

Question 3

(a) Rewrite the system as
$$\begin{cases} Q_r \quad\quad + 0.5p_r - 0.2p_a &= 12 \\ Q_r \quad\quad - 1.2p_r &= 0 \\ \quad\quad Q_a - 0.3p_r + 0.4p_a &= 9 \\ \quad\quad Q_a \quad\quad - 0.9p_a &= -3 \end{cases}$$

In matrix form, this can be formulated as
$$\begin{pmatrix} 1 & 0 & 0.5 & -0.2 \\ 1 & 0 & -1.2 & 0 \\ 0 & 1 & -0.3 & 0.4 \\ 0 & 1 & 0 & -0.9 \end{pmatrix} \begin{pmatrix} Q_r \\ Q_a \\ p_r \\ p_a \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 9 \\ -3 \end{pmatrix}$$

(b) $\mathbf{x}'$ will change into, for instance, $\mathbf{x}' = (Q_{rs}, Q_{rb}, Q_{as}, Q_{ab}, p_{rs}, p_{rb}, p_{as}, p_{ab})$. The order of $\mathbf{A}$ will become 8×8, and the order of $\mathbf{b}$ will become 8.

(c) In general, we can write the budget constraint as $f(R, W) = m$, and the maximum happiness as $H(R^*, W^*) = 7\left(R^*(m)\right)^{0.3}\left(W^*(m)\right)^{0.7}$, so as $H^*(m)$.
Lagrange's method implies that $\frac{dH^*(m)}{dm} = \lambda$.
In the original situation, the happiness is $H(15,35) = H^*(250) = 190$.
In the new situation (for $m = 252$), the happiness can be approximated by
$H^*(252) \approx H^*(250) + \left.\frac{dH^*(m)}{dm}\right|_{m=250} (252 - 250) = 190 + 2\lambda = 190 + 1.5 = 191.5$

(d) It is known that $f(T_F) = a + bT_F$, and that $T_C = \frac{5}{9}(T_F - 32)$, so we find $T_F = \frac{9}{5}T_C + 32$.
Further, $S = f(T_F) = g(T_C)$, because a change of temperature scale should not affect the sales S.
So $S = a + bT_F = a + b\left(\frac{9}{5}T_C + 32\right) = a + 32b + \frac{9}{5}bT_C = c + dT_C$.
Therefore, $c = a + 32b$ and $d = \frac{9}{5}b$.
The unit of c is dollar/capita, the unit of d is dollar/(capita×degree Celsius).