Business Mathematics (BK/IBA) – Quantitative Research Methods 1 (EBE)
Written exam – Full solutions 27 March 2015

Question 1

(a) Impossible

$\mathbf{A}^{-1}$ does not exist because $\mathbf{A}$ is not square.

(b) $4e^3$

$\frac{\partial\left(e^{p^2+pq}\right)}{\partial p} = \left(e^{p^2+pq}\right)\frac{\partial(p^2+pq)}{\partial p} = \left(e^{p^2+pq}\right)(2p+q)$.
At $(p,q) = (1,2)$, this gives $\left(e^{1^2+1\times 2}\right)(2\times 1+2) = (e^3)(4) = 4e^3$.

(c) 26

In general, $b_i = \sum_j a_{ij}x_j$, so $b_2 = \sum_j a_{2j}b_j$.
In this case, $b_2 = (3\times 3) + (-1\times -1) + (2\times 8) = 9 + 1 + 16 = 26$.

(d) $3x - x^2$

$\int_0^x (3-2y)dy = \int_0^x 3dy + \int_0^x -2ydy = [3y]_0^x - [y^2]_0^x = (3x-0) - (x^2-0) = 3x - x^2$

(e) $\left(\frac{3}{2}, 1\right)$

$g(x) = e^{(2x-3)^2}$ has stationary points when $\frac{dg}{dx} = 2(2x-3)2e^{(2x-3)^2} = 0$. This happens only when $2x-3=0$, so when $x = \frac{3}{2}$. $g\left(\frac{3}{2}\right) = e^{0^2} = e^0 = 1$. It is given that this is a minimum.

(f) (E)

$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}dx = 1$, therefore $C = \int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}dx = \frac{1}{\left(\frac{1}{\sqrt{2\pi}}\right)} = \sqrt{2\pi}$. This is approximately 2.507, so larger than 1 but smaller than 10.

(g) $y - x$

$\ln\frac{e^x e^y}{e^{2x}} = \ln\frac{e^y}{e^x} = \ln e^y - \ln e^x = y - x$

(h) 30

$S(3) = \sum_{i=3}^{2\times 3}(3+i) = (3+3) + (3+4) + (3+5) + (3+6) = 6 + 7 + 8 + 9 = 30$

(i) (G) or (E)

The idea was (E): to find the area A, we must let the horizontal coordinate run from left to right, so from $x = 0$ to $x = 100\%$. In this interval, we must consider the difference between the straight line and the curved line, so $x - g(x)$. The correct expression is therefore $A = \int_0^{100\%}\big(x - g(x)\big)dx$. However, in a last-minute change, g in the questions was changed into f, so the correct answer is (G). We still accept (E).

(i) (C)

Observe that $\sum_{i=1}^{n} x_i > 0$ while $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i < 0$. This can only happen when $n < 0$. But that is obviously absurd.

(k) 4.6

$\ln y = 38.1 + 4.6 \times \ln x \Leftrightarrow e^{\ln y} = e^{38.1+4.6\times\ln x} \Leftrightarrow$
$y = e^{38.1} \times e^{\ln x \times 4.6} = e^{38.1} \times \left(e^{\ln x}\right)^{4.6} = e^{38.1} \times x^{4.6} = a \times x^b \Rightarrow b = 4.6$

(l) 0, 4, and -2

$q^3 - 2q^2 = 8q \Leftrightarrow q(q^2 - 2q - 8) = 0 \Leftrightarrow q(q-4)(q+2) = 0 \Rightarrow q = 0 \vee q = 4 \vee q = -2$

(m) 378×10^8 or 37.8×10^9 or 3.78×10^{10} (or 377×10^8, etc)

$1500 \times 10^6 + 2.5 \times 10^8 + 3.5 \times 10^{10} + 10^9 = (1.5 + 0.25 + 35 + 1) \times 10^9 = 37.75 \times 10^9$

(n) $[-1,1]$ or $-1 \le p \le 1$

$\sqrt{1-p^2}$ exists when $1 - p^2 \ge 0$, so when $p^2 \le 1$. This means that $p \in [-1,1]$.

Question 2

(a) Element h_{ij} represents the how much the use of 1 minute of machine i improves physical aspect j. The elements of vector $\mathbf{t}$ represent the time spent on each machine, so t_i is the time (in minutes) spent on machine i. The elements of vector $\mathbf{p}$ represent the improvement of each physical aspect, so p_j is the improvement of physical aspect j. The total improvement of physical aspect j by employing all 17 machines for a specified duration is therefore $p_j = \sum_{i=1}^{17} h_{ij} t_i$ or $p_j = \sum_{i=1}^{17} t_i h_{ij}$. For all physical aspects, we can thus write $\mathbf{p} = \mathbf{H}'\mathbf{t}$ or $\mathbf{p} = \mathbf{t}'\mathbf{H}$.

(b') The objective is to maximize $f = (t_1)^{0.8}(t_2)^{0.4}$, under the constraint $t_1 + t_2 = 30$.
Thus, define a Lagrangian $\mathcal{L}(t_1, t_2, \lambda) = (t_1)^{0.8}(t_2)^{0.4} - \lambda(t_1 + t_2 - 30)$.
Possible stationary points of $\mathcal{L}$ occur when all three partial derivatives are 0:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial t_1} = 0.8 \times (t_1)^{-0.2}(t_2)^{0.4} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial t_2} = 0.4 \times (t_1)^{0.8}(t_2)^{-0.6} - \lambda = 0 \Rightarrow \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -t_1 - t_2 + 30 = 0 \end{cases}$$

$$\begin{cases} 0.8 \times (t_1)^{-0.2}(t_2)^{0.4} = \lambda \\ 0.4 \times (t_1)^{0.8}(t_2)^{-0.6} = \lambda \Rightarrow \\ -t_1 - t_2 + 30 = 0 \end{cases}$$

$$\begin{cases} 0.8 \times (t_1)^{-0.2}(t_2)^{0.4} = 0.4 \times (t_1)^{0.8}(t_2)^{-0.6} \\ t_1 = 30 - t_2 \end{cases} \Rightarrow$$

$$\begin{cases} 0.8 t_2 = 0.4 t_1 \\ t_1 = 30 - t_2 \end{cases} \Rightarrow$$

$$\begin{cases} 2t_2 = t_1 \\ 2t_2 = 30 - t_2 \end{cases} \Rightarrow$$

$$\begin{cases} t_1 = 20 \\ t_2 = 10 \end{cases}$$

So the salesman should use machine 1 for 20 minutes, and machine 2 for 10 minutes.

(b) The objective is to maximize $f = (12t_1 + 18t_2)^{0.8}(8t_1 + 6t_2)^{0.4}$, under the constraint $t_1 + t_2 = 30$.
Thus, define a Lagrangian $\mathcal{L}(t_1, t_2, \lambda) = (12t_1 + 18t_2)^{0.8}(8t_1 + 6t_2)^{0.4} - \lambda(t_1 + t_2 - 30)$.
Possible stationary points of $\mathcal{L}$ occur when all three partial derivatives are 0:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial t_1} = \cdots = 0 \\ \frac{\partial \mathcal{L}}{\partial t_2} = \cdots = 0 \quad \Rightarrow \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -t_1 - t_2 + 30 = 0 \end{cases}$$

The expression at the dots ... is rather long (it involves the product rule and the chain rule a couple of times), and solving the system of equations is even harder ...
It is easier to substitute the constraint in the fitness function:
maximize $f^*(t_1) = \left(12t_1 + 18(30 - t_1)\right)^{0.8}\left(8t_1 + 6(30 - t_1)\right)^{0.4} = (540 - 6t_1)^{0.8}(180 + 2t_1)^{0.4}$.
To do so find the derivative:

$$\frac{df^*}{dt_1} = 0.8 \times (540 - 6t_1)^{-0.2}(180 + 2t_1)^{0.4} \times -6 + (540 - 6t_1)^{0.8} \times 0.4 \times (180 + 2t_1)^{-0.6} \times 2$$

This should be equal to 0. So
$4.8 \times (540 - 6t_1)^{-0.2}(180 + 2t_1)^{0.4} = 0.8 \times (540 - 6t_1)^{0.8}(180 + 2t_1)^{-0.6} \Rightarrow$
$6 \times (180 + 2t_1) = (540 - 6t_1) \Rightarrow$
$1080 + 12t_1 = 540 - 6t_1 \Rightarrow$

$18t_1 = -540 \Rightarrow t_1 = -30$, and therefore $t_2 = 60$.
This is not a realistic solution.

(c) $M = \frac{165}{2.2} = 75$ kg, $H = 71 \times \frac{2.54}{100} = 1.8034$ m, $BMI = \frac{75}{1.8034^2} = 23.06095$, so the discount is 23.06095 euro, so $23.06095 \times 1.30 = 29.97$ dollar.

(d) $c = 0.25\, t\, w^{1.5}$, so $t = \frac{c}{0.25\, w^{1.5}} = 4\, c\, w^{-\frac{3}{2}}$. With $c = 128$ calories and $w = 64$ kg, this yields $t = 4 \times 128 \times 64^{-\frac{3}{2}} = 512 \times 8^{-3} = 1$ minute.

Question 3

(a) $\pi(s,r) = 2.00 \times s + 0.10 \times r - 1.20 \times (s + r) = 0.8 \times s - 1.1 \times r$

(b) To prove that p decreases when q increases, we study $\frac{dp}{dq}$. We have $\frac{dp}{dq} = -k \times 1.30 \times e^{-kq}$. This is a product of three terms:
i) $-k$, which is negative
ii) 1.30, which is positive
iii) e^{-kq}, which is positive
The product of two positive and one negative factors is always negative. Therefore the function $p(q)$ must be decreasing.

(c) $C(A,B) = A^2 + 3B^2 - 3AB - 2A + 12$. A stationary point requires both partial derivatives to be zero.

$$\begin{cases} \frac{\partial C}{\partial A} = 2A - 3B - 2 = 0 \\ \frac{\partial C}{\partial B} = 6B - 3A = 0 \end{cases} \Rightarrow$$

$$\begin{cases} 4B - 3B - 2 = 0 \\ 2B = A \end{cases} \Rightarrow$$

$$\begin{cases} B = 2 \\ 2B = A \end{cases} \Rightarrow$$

$$\begin{cases} B = 2 \\ A = 4 \end{cases}$$

To study the nature of this point, determine $\left(\frac{\partial^2 C}{\partial A^2}\right)\left(\frac{\partial^2 C}{\partial B^2}\right) - \left(\frac{\partial^2 C}{\partial A \partial B}\right)^2$.

We have $\left(\frac{\partial^2 C}{\partial A^2}\right) = 2$, $\left(\frac{\partial^2 C}{\partial B^2}\right) = 6$, and $\left(\frac{\partial^2 C}{\partial A \partial B}\right) = -3$. So, $\left(\frac{\partial^2 C}{\partial A^2}\right)\left(\frac{\partial^2 C}{\partial B^2}\right) - \left(\frac{\partial^2 C}{\partial A \partial B}\right)^2 = 12 - 9 = 3 > 0$.
Because this number is positive, the stationary point is an extreme value.
To investigate if the extreme value is a minimum or a maximum, consider $\left(\frac{\partial^2 C}{\partial A^2}\right) = 2 > 0$.
Therefore, the stationary point is a minimum.

(d) Rewrite the system as

$$\begin{cases} & 5B & -3C & -8\lambda & & = & 3 \\ 4A & +6B & & & +2\mu & = & -7 \\ & & & -9\lambda & +3\mu & = & -1 \\ & -2B & +4C & & & = & 0 \\ 2A & -7B & & & & = & 0 \end{cases}$$

In matrix form, this can be formulated as

$$\begin{pmatrix} 0 & 5 & -3 & -8 & 0 \\ 4 & 6 & 0 & 0 & 2 \\ 0 & 0 & 0 & -9 & 3 \\ 0 & -2 & 4 & 0 & 0 \\ 2 & -7 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

So, $\mathbf{A} = \begin{pmatrix} 0 & 5 & -3 & -8 & 0 \\ 4 & 6 & 0 & 0 & 2 \\ 0 & 0 & 0 & -9 & 3 \\ 0 & -2 & 4 & 0 & 0 \\ 2 & -7 & 0 & 0 & 0 \end{pmatrix}$; $\mathbf{b} = \begin{pmatrix} 3 \\ -7 \\ -1 \\ 0 \\ 0 \end{pmatrix}$; $\mathbf{x} = \begin{pmatrix} A \\ B \\ C \\ \lambda \\ \mu \end{pmatrix}$