# Business Mathematics (BK/IBA) – Quantitative Research Methods 1 (EBE)
# Written exam – Full solutions 23 April 2015

**Question 1**

(a) $\frac{2x-4y}{4x+4y}$

$\frac{d}{dx}(x^2-4xy-2y^2)=\frac{d}{dx}(6)\Rightarrow 2x-4y-4x\frac{dy}{dx}-4y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{2x-4y}{4x+4y}$

(b) $(\mathbf{C}'+\mathbf{B}')\mathbf{A}'$ or $\mathbf{C}'\mathbf{A}'+\mathbf{B}'\mathbf{A}'$

$\mathbf{A}^{-1}\mathbf{X}'-\mathbf{B}=\mathbf{C}\Rightarrow\mathbf{A}^{-1}\mathbf{X}'=(\mathbf{C}+\mathbf{B})\Rightarrow\mathbf{A}\mathbf{A}^{-1}\mathbf{X}'=\mathbf{A}(\mathbf{C}+\mathbf{B})\Rightarrow\mathbf{X}'=\mathbf{A}(\mathbf{C}+\mathbf{B})\Rightarrow$
$\mathbf{X}=\left(\mathbf{A}(\mathbf{C}+\mathbf{B})\right)'=(\mathbf{C}'+\mathbf{B}')\mathbf{A}'$

(c) (G)

$CV=\frac{s}{\bar{x}}$. $s\geq 0$, and $\bar{x}$ can have any value (except 0, in which case $CV$ is not defined). $CV$ therefore can run from $-\infty$ to $\infty$.

(d) $\frac{-1}{x}+\ln|x|+C$

$\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx=\int\frac{1}{x^2}dx+\int\frac{1}{x}dx=\frac{-1}{x}+\ln|x|+C$

(e) (C)

Mind that you should read the question carefully: What can you conclude? This means you should not try which of the answers (A)-(F) satisfies $\frac{df}{dx}=\frac{dg}{dx}$. For instance, (A) satisfies $\frac{df}{dx}=\frac{dg}{dx}$, but you cannot conclude (A) from $\frac{df}{dx}=\frac{dg}{dx}$.

(f) $2<x<4$ or $(2,4)$ or $\langle 2,4\rangle$

If $(x-2)>0$: $2>(x-2)\Rightarrow x<4\Rightarrow 2<x<4$
If $(x-2)<0$: $2<(x-2)\Rightarrow x>4\Rightarrow$contradiction

(g) $\mathbf{x}=\begin{pmatrix}12\\4\end{pmatrix}$ of $x_1=12$ and $x_2=4$ or $\mathbf{x}'=(12,4)$

$\left(\begin{array}{cc|c}-1&2&-4\\0&3&12\end{array}\right)\sim\left(\begin{array}{cc|c}-1&2&-4\\0&1&4\end{array}\right)\sim\left(\begin{array}{cc|c}-1&0&-12\\0&1&4\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&12\\0&1&4\end{array}\right)\Rightarrow\mathbf{x}=\begin{pmatrix}12\\4\end{pmatrix}$

(h) $\frac{-z}{(y-z)^2}$

$g(y,z)=\frac{z}{y-z}=z(y-z)^{-1}$, so $\frac{\partial g}{\partial y}=\frac{z}{(y-z)^2}\times -1$

(i) euro/km$^2$

The unit of $\gamma$ is the unit of $C$ divided by the unit of $d^2$. This yields the unit euro/km$^2$.

(k) $x_{10} - x_0$

$\sum_{i=1}^{10} x_i - \sum_{i=0}^{9} x_i = x_1 + x_2 + \cdots + x_9 + x_{10} - x_0 - x_1 - \cdots - x_9 = x_{10} - x_0$

(l) $c \geq -7$

$f(x,y) = x^2 + 2x^2y^2 + y^4 - 7 \geq -7$ because $x^2 + 2x^2y^2 + y^4$ has an obvious minimum of 0 (at $(x,y) = (0,0)$). So we can make level curves for $f(x,y) = c$ for $c \geq -7$.

(m) $\frac{1}{32}$

$f(x) = \sqrt{\sqrt{x}} = x^{\frac{1}{4}} \Rightarrow f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$.
At $x = 16$, this yields $f'(16) = \frac{1}{4}16^{-\frac{3}{4}} = \frac{1}{4} \times \frac{1}{8} = \frac{1}{32} (= 0.03125)$

(n) $(-3,-13)$ or $(-13,-3)$

$\begin{cases} xy = 39 \\ x + y = -16 \end{cases} \Rightarrow \begin{cases} xy = 39 \\ x = -16 - y \end{cases} \Rightarrow \begin{cases} (-16-y)y = 39 \\ x = -16 - y \end{cases}$.
This gives $-16y - y^2 = 39 \Rightarrow y^2 + 16y + 39 = 0 \Rightarrow (y+13)(y+3) = 0 \Rightarrow y = -13 \vee y = -3$.
The two pairs of solutions are therefore $(-3,-13)$ and $(-13,-3)$.

**Question 2**

(a) Requirement 2 implies: $\frac{dC}{dt} = abe^{bt} > 0$. Because $e^{bt} > 0$, we find that $ab > 0$.
Requirement 3 implies: $\frac{d^2C}{dt^2} = ab^2e^{bt} > 0$. Because $e^{bt} > 0$, we find that $ab^2 > 0$. So $a > 0$.
Requirements 2 and 3 together thus give $a > 0$ and $b > 0$.
This is fully compatible with requirement 1, because $e^{bt} > 1$ when $b > 0$ and $t \geq 0$.
So $a > 0$ and $b > 0$ (alternatively: $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$ or $a \in (0, \infty)$ and $b \in (0, \infty)$)

(b) Break-even implies $K = C = 5.3(e^{12.1t} - 1)$. This gives $\frac{K}{5.3} + 1 = e^{12.1t}$, so $12.1t = \ln\left(\frac{K}{5.3} + 1\right)$.
Hence, $t$ is given by the equation $t = \frac{1}{12.1}\ln\left(\frac{K}{5.3} + 1\right)$.
You may also write this as $t = 0.0826\ln(0.19K + 1)$.

(c) $R$ has stationary points when $\begin{cases} \frac{\partial R}{\partial D} = 4D - 4B = 0 \\ \frac{\partial R}{\partial B} = -4D + 4B^3 = 0 \end{cases} \Rightarrow \begin{cases} D = B \\ B = B^3 \end{cases} \Rightarrow B = 0 \vee B = 1 \vee B = -1.$
So the stationary points are $(0,0)$, $(1,1)$, and $(-1,-1)$.
Only $(1,1)$ is within the domain for further analysis.
To investigate the nature of these two points, consider the second derivatives.
$\frac{\partial^2 R}{\partial D^2} = 4$, $\frac{\partial^2 R}{\partial B^2} = 12B^2$, $\frac{\partial^2 R}{\partial B \partial D} = -4$, so $\frac{\partial^2 R}{\partial D^2}\frac{\partial^2 R}{\partial B^2} - \left(\frac{\partial^2 R}{\partial B \partial D}\right)^2 = 48B^2 - 16$
For $(1,1)$, this gives 32 so the stationary point is an extremum. Because $\frac{\partial^2 R}{\partial D^2} = 4 > 0$, it is a minimum.
The value in this minimum point is $R = 2 - 4 + 1 + 2 = 1$.

(d) In general, $r_{\mathbf{x},\mathbf{y}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$. In our case, $n = 3000$, $x_i = p_i$, and $y_i = b_{270+i}$, where the shift of 270 is made to introduce the time lag of 9 months. (We accept other values around 270 because 1 month is not exactly 30 days).
Filling in all we have, the expression becomes $r = \frac{\sum_{i=1}^{2730}(p_i - \bar{p})(b_{270+i} - \bar{b})}{\sqrt{\sum_{i=1}^{2730}(p_i - \bar{p})^2 \sum_{i=1}^{2730}(b_{270+i} - \bar{b})^2}}$.
Notice that the sum runs until $3000 - 270 = 2730$, because otherwise we would run out of the data set of **b**.

**Question 3**

(a) $g(h,u) = h^{0.3}u^{0.2}$, so $\frac{\partial g}{\partial h} = 0.3h^{-0.7}u^{0.2}$. For all choices of $h$ and $u$, this partial derivative is positive. So whenever $h$ increases, $g$ increases.

(b) Define a Lagrangian $\mathcal{L}(h,u,\lambda) = h^{0.3}u^{0.2} - \lambda(h + u - 150)$. Then a possible constrained maximum of $g$ occurs when $\mathcal{L}$ has a stationary point.

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial h} = 0.3h^{-0.7}u^{0.2} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial u} = 0.2h^{0.3}u^{-0.8} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -(h + u - 150) = 0 \end{cases} \Rightarrow \begin{cases} 0.3h^{-0.7}u^{0.2} = 0.2h^{0.3}u^{-0.8} \\ h + u = 150 \end{cases} \Rightarrow \begin{cases} 0.3u = 0.2h \\ h + u = 150 \end{cases} \Rightarrow \begin{cases} u = 60 \\ h = 90 \end{cases}$$

So the student should study 60 hours in university class, and 90 hours at home.
His grade will be $g = 60^{0.3} \times 90^{0.2} \approx 8.8$.

(c) Define $g^*(h) = h^{0.3}48^{0.2}$. It is given that $g^*(120) = 9.1$.
A first-order approximation of $g^*$ around $h = 120$ is given by

$$g^*(h) \approx g^*(120) + \left.\frac{dg^*}{dh}\right|_{h=120}(h - 120) = 9.1 + 0.3 \times 120^{-0.7} \times 48^{0.2}(h - 120).$$

At $h = 121$, this gives $g^*(121) \approx 9.1 + 0.3 \times 120^{-0.7} \times 48^{0.2} \approx 9.1 + 0.0228 = 9.1228 \approx 9.12$

Note that the approach using $\lambda$ from question (b) is wrong. It is tempting to use $\frac{dg^*}{dh} = \lambda$, but the value of $\lambda$ you might wish to take from (b) is only applicable for the case $(h,u) = (90,60)$, while we need to develop the linear approximation around $(h,u) = (120,48)$ in (c).