# Business Mathematics (BK/IBA) – Quantitative Research Methods 1 (EBE)
# Written exam – Full solutions XX October 2014

**Question 1**

(a) $0.6\ \frac{€}{\text{cm}^2}$
because $5\ \frac{\$}{\text{inch}^2} = 5\ \frac{0.77\ €}{(2.54\ \text{cm})^2} = 5 \times \frac{0.77}{2.54^2}\ \frac{€}{\text{cm}^2} = 0.5967\frac{€}{\text{cm}^2}$

(b) $11$
because $\sum_{i=2}^{3} x_{i,i-1} = x_{21} + x_{32} = 9 + 2 = 11$

(c) $[0,1) \cup (1, \infty)$
because (1) the square root may not be negative, so $x \geq 0$, and (2) the denominator may not be zero, so $x \neq 1$.

(d) $-\frac{y+2xy^2}{x+2yx^2}$
because $yx + y^2x^2 = 12 \Rightarrow \frac{d}{dx}(yx + y^2x^2) = \frac{d}{dx}(12) \Rightarrow y'x + y + 2yy'x^2 + y^2 \cdot 2x = 0 \Rightarrow y'(x + 2yx^2) = -y - 2xy^2 \Rightarrow y' = -\frac{y+2xy^2}{x+2yx^2}$

(e) $40.7$
becasue $\int_1^5 \left(\frac{1}{2}x^2 + 5\right) dx = \frac{1}{2}\int_1^5 x^2 dx + \int_1^5 5dx = \left[\frac{1}{6}x^3 + 5x\right]_1^5 = \frac{1}{6}5^3 + 5 \cdot 5 - \left(\frac{1}{6}1^3 + 5 \cdot 1\right) = \frac{125}{6} + 25 - \frac{1}{6} - 5 = \frac{124}{6} + 20 = 40.6667$

(f) $10$
becasue, $\sqrt{3^{z-2}} = 81 \Rightarrow 3^{z-2} = 81^2 = (3^4)^2 \Rightarrow z - 2 = 8 \Rightarrow z = 10$

(g) $(\mathbf{A}^{-1})'(\mathbf{C}' + \mathbf{B}')$ or $(\mathbf{A}')^{-1}(\mathbf{C}' + \mathbf{B}')$
because, $\mathbf{Q}'\mathbf{A} - \mathbf{B} = \mathbf{C} \Rightarrow \mathbf{Q}'\mathbf{A} = \mathbf{C} + \mathbf{B} \Rightarrow \mathbf{Q}' = (\mathbf{C} + \mathbf{B})\mathbf{A}^{-1} \Rightarrow \mathbf{Q} = \left((\mathbf{C} + \mathbf{B})\mathbf{A}^{-1}\right)' = (\mathbf{A}^{-1})'(\mathbf{C} + \mathbf{B})' = (\mathbf{A}')^{-1}(\mathbf{C}' + \mathbf{B}')$

(h) $-\frac{x^3}{(y-1)^2}$
because $f(x, y) = \frac{x^3}{y-1} \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^3}{(y-1)^2}$ (chain rule)

(i) $4.0 \cdot 10^3$ or $40 \cdot 10^2$
because $n = 23$, $\bar{x} = 5.2$, $CV_x = 12.2$, and $s_x^2 = (CV_x\bar{x})^2$, so the variance of $\mathbf{x}$ is $(12.2 \times 5.2)^2 = 4024.634$

(j) $(3 \times 3)$
because the order of $\mathbf{AB}'\mathbf{A}'$ is $(3 \times 4) \times (4 \times 4) \times (4 \times 3) = (3 \times 3)$

(k) (D)

(l) $2^{99}e^{2x}$

because $f(x) = e^{2x} + x^6 + 5 \Rightarrow f'(x) = 2e^{2x} + 6x^5 \Rightarrow f''(x) = 2^2 e^{2x} + 30x^4 \Rightarrow f'''(x) = 2^3 e^{2x} + 120x^3 \Rightarrow \cdots \Rightarrow \frac{d^{99} f(x)}{dx^{99}} = 2^{99} e^{2x}$

(m) $\frac{1}{2}e^{2t} - \frac{4}{3}t^{\frac{3}{2}} + C$

because $\int\left(e^{2t} - \sqrt{4t}\right)dt = \int e^{2t}dt - \int 2\sqrt{t}dt = \frac{1}{2}e^{2t} + C_1 - \frac{2}{3}2t^{\frac{3}{2}} - C_2 = \frac{1}{2}e^{2t} - \frac{4}{3}t^{\frac{3}{2}} + C$

(n)
$$\begin{cases} \text{maximize} & e^{x_1+x_2} \\ \text{subject to} & 2x_1 + x_2 \leq 8 \\ \text{and} & x_1 + x_2 \leq 9 \\ \text{and} & 3x_1 + 2x_2 \leq 24 \\ \text{and} & x_1 \geq 0 \\ \text{and} & x_2 \geq 0 \end{cases}$$

becasue (1) the objective is stated as $e^{x+_1x_2}$ (2) line $s_1$ is given by $x_2 = 8 - \frac{1}{2}x_1$ (3) line $s_2$ is given by $x_2 = 9 - x_1$ (4) line $s_3$ is given by $x_2 = 12 - \frac{3}{2}x_1$ and (5) the horizontal and vertical axes give two further non-negativity constraints.

**Question 2**

(a) $C = aL^{0.2}M^{0.8}$. The necessary first-order condition for a stationary point is (1) $\frac{\partial C}{\partial L} = 0$ and (2) $\frac{\partial C}{\partial M} = 0$.

(1) $\frac{\partial C}{\partial L} = 0.2aL^{-0.8}M^{0.8} = 0$ and (2) $\frac{\partial C}{\partial M} = 0.8aL^{0.2}M^{-0.2} = 0$.

Both equations have no solutions for $L$ and $M$. Hence there is no stationary point of $C$.

(b) Demand: $Q = \alpha - \beta P^{\gamma}$, supply $Q = \delta + \epsilon P^{\gamma}$. Equilbrium is a point $(P^*, Q^*)$ where both equations hold.

$$\begin{cases} Q^* = \alpha - \beta {P^*}^{\gamma} \\ Q^* = \delta + \epsilon {P^*}^{\gamma} \end{cases} \Rightarrow \alpha - \beta {P^*}^{\gamma} = \delta + \epsilon {P^*}^{\gamma} \Rightarrow \alpha - \delta = \beta {P^*}^{\gamma} + \epsilon {P^*}^{\gamma} \Rightarrow {P^*}^{\gamma} = \frac{\alpha-\delta}{\beta+\epsilon} \Rightarrow$$

$$P^* = \left(\frac{\alpha-\delta}{\beta+\epsilon}\right)^{1/\gamma}$$

(c) Demand as a function of $\beta$ and the other parameters: $Q(\beta) = \alpha - \beta P^{\gamma}$.

Elasticity: $\text{El}_{\beta}Q(\beta) = \frac{\beta}{Q(\beta)}Q'(\beta)$. Obviously $Q'(\beta) = -P^{\gamma}$, so $\text{El}_{\beta}Q(\beta) = \frac{-\beta}{\alpha-\beta P^{\gamma}}P^{\gamma}$.

Note that an even more correct form is $Q(\alpha, \beta, \gamma) = \alpha - \beta P^{\gamma}$. With that, $Q'(\beta)$ must be replaced by $\frac{\partial Q(\alpha,\beta,\gamma)}{\partial \beta}$.

(d) $\begin{cases} \max bM\sqrt{L} \\ \text{subject to } L + M = 600 \end{cases}$. The Lagrangian is $\mathcal{L}(L, M, \lambda) = bM\sqrt{L} - \lambda(L + M - 600)$.

Stationary points: $\begin{cases} \frac{\partial \mathcal{L}}{\partial L} = b\frac{M}{2\sqrt{L}} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial M} = b\sqrt{L} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -L - M + 600 = 0 \end{cases}$

The first two equations yield $b\frac{M}{2\sqrt{L}} = b\sqrt{L} \Rightarrow M = 2L$.

Inserted in the constraint $L + M = 600$, we have $L + 2L = 600 \Rightarrow L = 200$ (and $M = 400$).

To find if this is a maximum: take two points on two sides of the constraint, e.g., $(600{,}0)$ and $(0{,}600)$. At these values of $M$ and $L$ the profit is $0$, which is lower than the value at $(200{,}400)$ (Remember that $b > 0$).

**Question 3**

(a) $S_C = \int_0^{Q^*} (D(Q) - P^*) dQ = \int_0^{Q^*} D(Q) dQ - P^* Q^*$

(b) The producer surplus is a quantity multiplied by a price, so kg divided by kg/€, so it is in €.

(c) The number of products sold $Q$ is a function of the price $P$, so $Q = Q(P)$. The rate of change of $Q$ is $\frac{dQ}{dP}$.

(d) $$\begin{cases} Q_1 = 5 - 8P_1 + 2P_2 \\ Q_2 = 12 - 4P_1 + 3P_2 \\ Q_1 = 7 + 4P_1 \\ Q_2 = 4 + 9P_2 \end{cases} \Rightarrow \begin{cases} 8P_1 & + & Q_1 & + & -2P_2 & & & = & 5 \\ 4P_1 & & & + & -3P_2 & + & Q_2 & = & 12 \\ -4P_1 & + & Q_1 & & & & & = & 7 \\ & & & & -9P_2 & + & Q_2 & = & 4 \end{cases} \Rightarrow$$

$$\begin{pmatrix} 8 & 1 & -2 & 0 \\ 4 & 0 & -3 & 1 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -9 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 7 \\ 4 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 8 & 1 & -2 & 0 \\ 4 & 0 & -3 & 1 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -9 & 1 \end{pmatrix} \wedge \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 7 \\ 4 \end{pmatrix}$$

(e) $\pi = 6P_1^2 - 3P_1P_2 + 3P_2^2 - 7P_1 + 112$ Stationary points require:

$$\begin{cases} \frac{\partial \pi}{\partial P_1} = 12P_1 - 3P_2 - 7 = 0 \\ \frac{\partial \pi}{\partial P_2} = -3P_1 + 6P_2 = 0 \end{cases} \xRightarrow{\text{multiply 2nd equation by 4 and add}} \begin{cases} P_2 = \frac{1}{3} \\ P_1 = \frac{2}{3} \end{cases}$$

So the only stationary point is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

The nature of this point is found by the second-order criterion:

$\frac{\partial^2 \pi}{\partial P_1^2} = 12$, $\frac{\partial^2 \pi}{\partial P_2^2} = 6$, and $\frac{\partial^2 \pi}{\partial P_1 \partial P_2} = -3$, so $\frac{\partial^2 \pi}{\partial P_1^2} \frac{\partial^2 \pi}{\partial P_2^2} - \left(\frac{\partial^2 \pi}{\partial P_1 \partial P_2}\right)^2 = 12 \cdot 6 - (-3)^2 = 72 - 9 = 63 > 0$, so the only stationary point is an extreme value.

Further, $\frac{\partial^2 \pi}{\partial P_1^2} = 12 > 0$, so it is a minimum.