The Max-Plus Algebra: A New Approach To Performance Evaluation of Discrete Event Systems

Bernd Heidergott

TU Eindhoven

Department of Mathematics and Computing Science

IFORS, July 8, 2002

Outline Of The Tutorial

- Semirings
- The (Max,+) Semiring
- Four Good Reasons for Working with the (Max,+) Semiring
- Max Plus at Work
  - The deterministic setup [Public Transportation]
  - Intermezzo: Max Plus Models [Queuing model]
  - Max Plus for Stochastic Systems [Heaps of pieces]
- Concluding remarks

Before we start, the basic reference (still is)

François Baccelli, Guy Cohen, Geert Jan Olsder and Jean–Pierre Quadrat
Synchronization and Linearity: An Algebra for Discrete Event Systems
John Wiley and Sons, New–York, 1992

This book is out of print, but it can be downloaded from the web via

http://www-rocq.inria.fr/scilab/cohen/SED/book-online.html

Semirings

Semiring: Definition

A semiring is a non–empty set R endowed with two binary relations $\oplus$ and $\otimes$ such that

- the operation $\oplus$ is associative, commutative, has zero element ϵ and $\otimes$ distributes over $\oplus$;
- the operation $\otimes$ is associative, has unit element e and ϵ is absorbing for $\otimes$:

$$\forall a \in R: \quad a \otimes \epsilon = \epsilon \, .$$

A semiring is denoted by $\mathcal{R} = (R, \oplus, \otimes, \epsilon, e)$.

Semiring: Definition (cont.)

A semiring is called idempotent if

$$\forall a \in R : \quad a \oplus a = a \, ,$$

and commutative if $\otimes$ is commutative.

Idempotent semirings are called dioids in BCOQ(1992).

Semiring: Examples

R	$\mathcal{P}(R)$	$\mathbb{R} \cup \{-\infty\}$	$\mathbb{R} \cup \{\infty\}$	$\mathbb{R}$
$\oplus$	$\cap$	max	min	$+$
ϵ	R	$-\infty$	∞	0
$\otimes$	$\cup$	$+$	$+$	$\times$
e	$\emptyset$	0	0	1
idemp.	$\ddot\smile$	$\ddot\smile$	$\ddot\smile$	

Idempotency

Idempotency of $\oplus$ rules out invertability of $\oplus$.

Proof: Suppose that for $a \neq \epsilon$ a number b exists such that

$$a \oplus b = \epsilon \,.$$

Adding a on both sides yields

$$a \oplus a \oplus b = a \oplus \epsilon \,.$$

By idempotency, this is equivalent to

$$a \oplus b = a \oplus \epsilon \,,$$

which implies

$$a \oplus b = a \,.$$

This contradicts $a \oplus b = \epsilon$.

Particular Semirings

The structure

$$\mathcal{R}_{\max} = (\mathbb{R}_\epsilon = \mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +, \epsilon = -\infty, e = 0)$$

constitutes an idempotent semiring known as (max,+)–algebra .

The structure

$$\mathcal{R}_{\min} = (\mathbb{R}_\top = \mathbb{R} \cup \{\infty\}, \oplus = \min, \otimes = +, \top = \infty, e = 0)$$

constitutes an idempotent semiring known as (min,+)–algebra . Note that for the (min,+) algebra the notation $\top$ for the zero element of $\oplus$ is standard.

Other Important Semirings

In network calculus one defines, for example, $(f \oplus g)(t) = \max(f(t), g(t))$ and

$$(f \otimes g)(t) = \sup_{0 \le s \le t} \Big(f(t-s) + g(s) \Big) .$$

Network Calculus at IFORS:

Session TD16, venue WR-11, ”On probabilistic network calculus”, M. Vojnovic and J.–Y. Le Boudec.

Session TD16, venue WR-11, ”Some results of deterministic network calculus applied to communication networks”, P. Thiran and J.–Y. Le Boudec.

More on semirings:
Stephané Gaubert. Methods and applications of (max,+)–linear algebra
In *Proccedings of the STACS'1997*, Lecture Notes in Computer Science,
vol 1200, Springer, 1997 (this report can be accessed via the WEB at
http://www.inria.fr/RRRT/RR-3088.html)

Matrices and Vectors in $\mathcal{R} = (R, \oplus, \otimes, \epsilon, e)$

For matrices $A \in R_\epsilon^{I \times K}$ and $B \in R_\epsilon^{K \times J}$ we define the matrix product $A \otimes B$ in the usual way:

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{K} A_{ik} \otimes B_{kj} \, .$$

Specifically, we introduce the i^{th} power of A by

$$A^i = \underbrace{A \otimes \ldots \otimes A}_{i \text{ times}} ,$$

where $A^0 = E$.

Addition of matrices $A \in R_\epsilon^{J \times I}$ and $B \in R_\epsilon^{J \times I}$, denoted by $A \oplus B$, is given by

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \, .$$

The (Max,+) Semiring

The (Max,+) Semiring: Basic Calculus

[Recall that $\mathcal{R}_{\mathsf{max}} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +, \epsilon = -\infty, e = 0)$]

$$5 \oplus 3 =$$

$$5 \oplus \epsilon =$$

$$e \oplus 3 =$$

$$4^3 = \left(4^{\otimes 3} = \right)$$

$$\sqrt[3]{9} =$$

$$\sqrt{-1} =$$

The (Max,+) Semiring: Matrix-Vector Calculus

$$\begin{pmatrix} e & \epsilon \\ 3 & 2 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 1 \end{pmatrix} =$$

The (Max,+) Semiring: Polynomials

Consider the polynomial

$$\left(x \oplus 1\right)^2$$

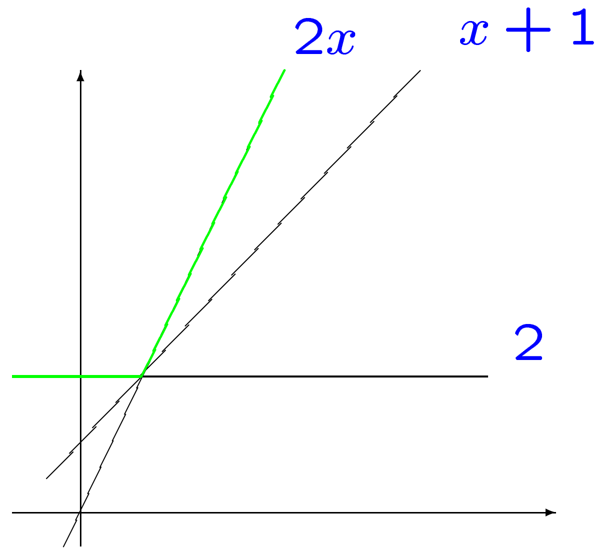
By algebraic computations,

$$\begin{aligned}\left(x \oplus 1\right)^2 &= \left(x \oplus 1\right) \otimes \left(x \oplus 1\right) \\ &= \left(x \otimes x\right) \oplus \left(x \otimes 1\right) \oplus \left(x \otimes 1\right) \oplus (1 \otimes 1) \\ &= x^2 \oplus \left(x \otimes 1\right) \oplus 2\end{aligned}$$

Numerically,

$$\begin{aligned}&= \max\left(2x\,,\, x+1\,,\, 2\right) \\ &= \max\left(2x\,,\, 2\right) \\ &= x^2 \oplus 2\,.\end{aligned}$$

The (Max,+) Semiring: Polynomials (cont.)



$$\left(x \oplus 1\right)^2 = x^2 \oplus \left(x \otimes 1\right) \oplus 2 = x^2 \oplus 2\,.$$

Generally,

$$\left(a \oplus b\right)^n = a^n \oplus a^{n-1}b \oplus \cdots \oplus ab^{n-1} \oplus b^n\,.$$

Warning! Hybrid Formulas

Let $A \in \mathbb{R}_\epsilon^{J \times J}$ and $x_0 \in \mathbb{R}_\epsilon^J$ and consider

$$x(k+1) = A \otimes x(k)\,, \quad k \geq 0$$
$$x(0) = x_0\,.$$

How do we interpret the following formula:

$$\lim_{k \to \infty} \frac{1}{k} x(k) = \lim_{k \to \infty} \frac{1}{k} A^k \otimes x_0 \quad ?$$

Four Good Reasons for Working with the (Max,+) Semiring

A First Good Reason for (Max,+): Computing Maximal Weights in Graphs

Let $A \in \mathbb{R}_{\epsilon}^{J \times J}$. The communication graph of A, denoted by $\mathcal{G}(A)$, is defined as follows.

$\mathcal{G}(A)$ has nodes $\{1, \ldots, J\}$, and a pair $(i, j) \in J \times J$ is an arc of the graph if $A_{ji} \neq \epsilon$.

For any arc (i, j) in $\mathcal{G}(A)$, we call A_{ji} the weight of arc (i, j) and the weight of a path in $\mathcal{G}(A)$ is defined by the sum of the weights of all arcs constituting the path.

Then, $(A^n)_{ji}$ yields the maximal weight of a path of length n (that is, consisting of n arcs) from node i to node j, and $(A^n)_{ji} = \epsilon$ refers to the fact that there is no path of length n from i to j, in $\mathcal{G}(A)$.

A Second Good Reason for (Max,+): Solving Linear Equations

Let $x, b \in \mathbb{R}_\epsilon^J$ and $A \in \mathbb{R}_\epsilon^{J \times J}$, solve

$$x = A \otimes x \oplus b \,. \tag{1}$$

We define the power series of A by

$$A^* = \bigoplus_{i=0}^{\infty} A^i \,,$$

which is finite if A is a lower triangular matrix.

Then, $x = A^* \otimes b$ solves (1).

Intermezzo: Irreducibility

A matrix $A \in \mathbb{R}_{\epsilon}^{J \times J}$ is called irreducible if its communication graph is strongly connected.

In words: for any two nodes i, j there exits a path in $\mathcal{G}(A)$.

A Third Good Reason for (Max,+): Eigenvalues and Eigenvectors

For any irreducible matrix $A \in \mathbb{R}_\epsilon^{J \times J}$, uniquely defined integers $c(A)$, $\sigma(A)$ and a uniquely defined real number $\lambda = \lambda(A)$ exist such that, for all $n \geq c(A)$:

$$A^{n+\sigma(A)} = \lambda^{\otimes \sigma(A)} \otimes A^n \, .$$

The number $c(A)$ is called the coupling time of A, $\sigma(A)$ is called the cyclicity of A and $\lambda^{\otimes \sigma(A)}$ is the unique eigenvalue of $A^{\otimes \sigma(A)}$.

This is also called the "Perron-Frobenius Theorem of (max,+) algebra".

λ is also called the Lyapunov exponent of A.

A Third Good Reason for (Max,+): Eigenvalues and Eigenvectors (cont.)

Let $A \in \mathbb{R}_\epsilon^{J \times J}$ and $x_0 \in \mathbb{R}_\epsilon^J$ and consider

$$\begin{aligned} x(k+1) &= A \otimes x(k)\,, \quad k \geq 0 \\ x(0) &= x_0\,. \end{aligned}$$

For all $k \geq c(A)$:

$$\begin{aligned} x(k+\sigma(A)) &= A^{k+\sigma(A)} \otimes x_0 \\ &= \lambda^{\otimes \sigma(A)} \otimes A^k \otimes x_0 \\ &= \lambda^{\otimes \sigma(A)} \otimes x(k)\,. \end{aligned}$$

We say that $\{x(k)\}$ enters its periodic regime after (at most) $c(A)$ transitions.

For any initial vector $x(0)$, the limiting behaviour of the sequence $\{x(k)\}$ is

$$\lim_{k \to \infty} \frac{x_j(k)}{k} = \lambda\,, \quad 1 \leq j \leq J\,.$$

A Third Good Reason for (Max,+): Eigenvalues and the Commun. Graph

Let $\mathcal{C}$ denote the set of circuits in $\mathcal{G}(A)$. It then holds that

$$\lambda = \max_{p \in \mathcal{C}} \frac{\text{weight of } p}{\text{length of } p}.$$

A circuit whose average weight is maximal (equals λ) is called cirtical.

The critical graph is the subgraph of $\mathcal{G}(A)$ that contains the critical circuits only.

The critical graph determines the cyclicity of A: If the critical graph is strongly connected, then the cyclicity of A is given by the greatest common divisor of the lengths of all circuits in the critical graph.

The critical graph characterizes the eigenspace of $A^{\otimes \sigma(A)}$.

A Third Good Reason for (Max,+): Computational Issues

- Power algorithm (eigenvalue and eigenvector)

- Karp's algorithm (eigenvalue)

- The Howard algorithm allows for computing the eigenvalue and an eigenvector in almost linear time (eigenvalue and eigenvector)

- Computing the coupling time is NP hard in the number of critical circuits, no efficient algorithms exist and only (crude) upper bounds are known

A Fourth Good Reason for (Max,+): Subadditivity

For $A \in \mathbb{R}_\epsilon^{J \times J}$, set

$$||A||_\vee = \bigoplus_{i=1}^{J} \bigoplus_{j=1}^{J} A_{ij} = \max\{ A_{ij} : 1 \leq i, j \leq J \} .$$

It then holds for $A, B \in \mathbb{R}_\epsilon^{J \times J}$ that

$$||A \otimes B||_\vee \leq ||A||_\vee + ||B||_\vee .$$

Let $A(k) \in \mathbb{R}_\epsilon^{J \times J}$, for $k \geq 0$, and set

$$\xi_{lk} = \left\| \bigotimes_{i=l}^{k-1} A(i) \right\|_\vee ,$$

then $\{\xi_{lk} : k \geq 1 ; 0 \leq l < k\}$ is subadditive, that is,

$$\xi_{lk} \leq \xi_{lm} + \xi_{mk} ,$$

for all m with $l < m < k$.

Max Plus at Work

(Max,+)-Linear Discrete Event Systems

Let $A \in \mathbb{R}_\epsilon^{J \times J}$ and consider the homogeneous recursion

$$\begin{aligned} x(k+1) &= A \otimes x(k)\,, \quad k \geq 0 \\ x(0) &= x_0\,. \end{aligned}$$

Alternatively, consider the inhomogeneous recursion

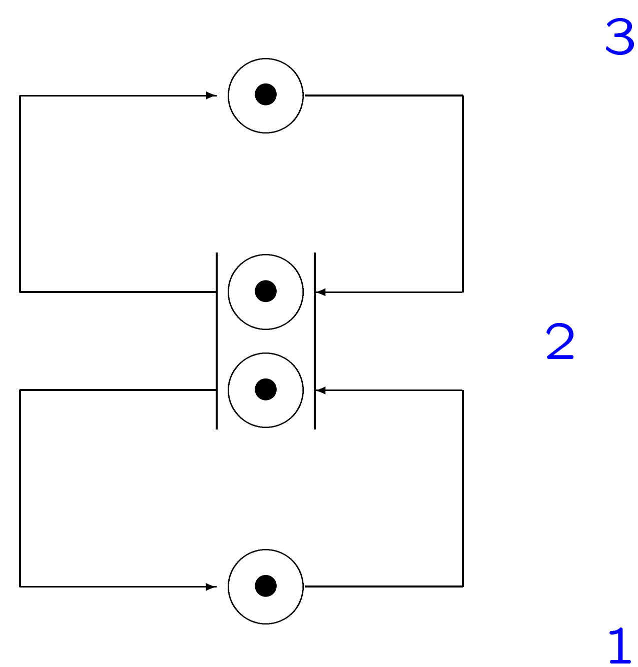
$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus b(k+1)\,, \quad k \geq 0 \\ x(0) &= x_0\,, \end{aligned}$$

with $b(k+1) \in \mathbb{R}_\epsilon^J$.

A system whose state–dynamic follows either of the above recursion is called (max,+)–linear.

(Max,+)–linear systems arise naturally in the presence of synchronization.

(Max,+)–Linear Systems: A Public Transportation Example



Four trains circulate on separated lines. At the center station, the departure of trains is synchronized in order to let passengers change trains.

A Public Transportation Example: The (Max,+) Model

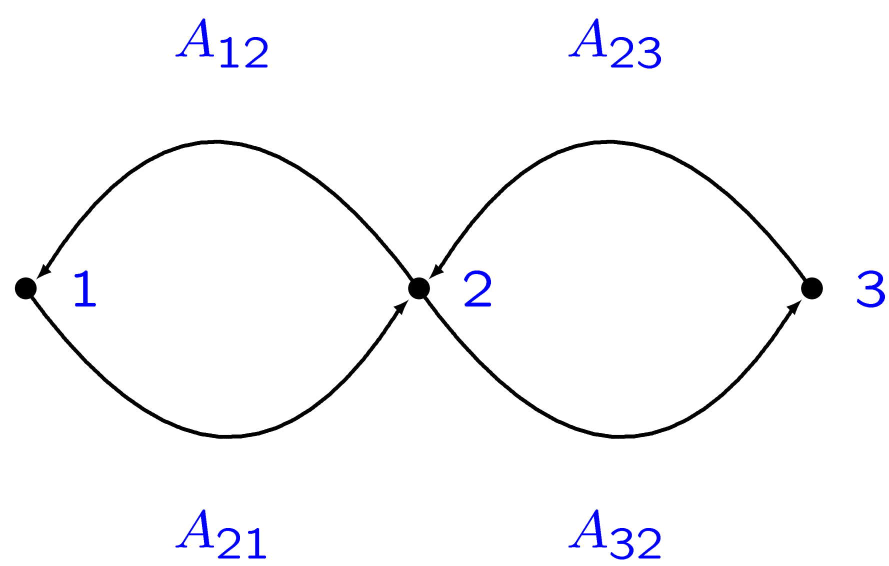
Let A_{ij} denote the travel time from station j to i (including dwell times) and let $x_j(k)$ denote the k^{th} departure time from station j, then

$$\begin{aligned} x_1(k+1) &= A_{12} \otimes x_2(k) \\ x_2(k+1) &= \big(A_{21} \otimes x_1(k)\big) \oplus \big(A_{23} \otimes x_3(k)\big) \\ x_3(k+1) &= A_{32} \otimes x_2(k) \,. \end{aligned}$$

In matrix–vector nation:

$$x(k+1) = \begin{pmatrix} \epsilon & A_{12} & \epsilon \\ A_{21} & \epsilon & A_{23} \\ \epsilon & A_{32} & \epsilon \end{pmatrix} \otimes x(k) \,.$$

A Public Transportation: Computing the Eigenvalue



If $A_{12} + A_{21} > A_{23} + A_{32}$, then $1 \to 2 \to 1$ is the critical circuit

$$\lambda = \frac{A_{12} + A_{21}}{2} \quad \text{and} \quad \sigma(A) = 2\,.$$

Timetable Design, I

Let A be a matrix of cyclicity 1 modeling the actual traveling times of trains on the tracks. Then, the vector of $(k+1)^{st}$ departure times per track follows

$$x(k+1) = A \otimes x(k)\,, \quad k \in \mathbb{N}\,,$$
$$x(0) = x_0\,.$$

Assume that, in addition to that, A is irreducible. By the (max,+) Perron-Frobenius theorem, for sufficiently large k,

$$x(k+1) = \lambda \otimes x(k)\,,$$

where λ denotes the eigenvalue of A.

Timetable Design, II

Let X denote an eigenvector of A and take $x_0 = X$, then

$$x(k+1) = A^k \otimes X = (k \cdot \lambda) \otimes X \,.$$

Hence, X represents a timetable and λ the frequency, or, speed of the timetable.

X is optimal in the sense that it represents the timetable with the highest frequency of trains per track physically possible.

For $\tau \geq \lambda$, let $d(k) = (k \cdot \tau) \otimes X$ denote the vector of planned k^{th} departure times per track according to timetable X, then the actual departure times $x(k)$ are given by

$$x(k+1) = A \otimes x(k) \oplus d(k+1) \,.$$

The difference $\tau - \lambda$ is an indicator for the robustness of the timetable.

Propagation of Delays, I

By simple algebra,

$$\begin{aligned}
x(k+1) &= A \otimes x(k) \oplus d(k+1) \\
&= A \otimes (A \otimes x(k-1) \oplus d(k)) \oplus d(k+1) \\
&= A^2 \otimes x(k-1) \oplus A \otimes d(k) \oplus d(k+1) \\
&= A^2 \otimes x(k-1) \oplus d(k+1) \\
&\vdots \\
&= A^k \otimes x(1) \oplus d(k+1)\,.
\end{aligned}$$

Let the first train which departs on track j be delayed, so that $x_j(1) > d_j(1)$, and assume that this is the only train that is delayed.

Propagation of Delays, II

The initial delay on track j causes a delay for the $(k+1)^{st}$ train departing on track i if

$$\bigoplus_{l} (A^k)_{i\,l} \otimes x_l(1) > d_i(k+1)\,.$$

Because of our assumption that $x_i(1) \le d_i(1)$, for $i \neq j$, it follows that the $(k+1)^{st}$ departure on track i is delayed because of a delay in the initial departure on track j if

$$(A^k)_{ij} \otimes x_j(1) > d_i(k+1)\,.$$

In this way we obtain the set of all delayed trains.

Propagation of Delays, III

Observe that the matrices A^k $(k = 1, 2, \ldots)$ can be calculated in advance and that to determine the propagation of an initial delay on track j, we only need the j^{th} column of these matrices.

After k^* steps, where k^* is given by

$$k^* = \min \left\{ k \,\middle|\, (A^k)_{ij} \otimes x_j(1) \leq d_i(k+1) \forall i \right\},$$

the initial delay on j is out of the system.

Propagation of Delays and the Coupling Time

Consider the system with initial vector X':

$$\begin{aligned} x'(k+1) &= A \otimes x'(k)\,, \quad k \in \mathbb{N}\,, \\ x'(0) &= X'\,. \end{aligned}$$

Recall that the coupling time of A is denoted by $c(A)$. Hence, independent of X', it holds that

$$x(k) = a \otimes x'(k)\,, \quad k \geq c(A)\,,$$

for some finite number a. (Here, we assume that A has an unique eigenvector.)

In words, a delay either dies out after at most $c(A)$ transitions or results in a uniform delay of a time units on all tracks: a is the part of the delay that reaches the critical circuit.

Applications of (max,+) to Railway Systems at IFORS

Semi-plenary, Section TA16, venue WR-11 ”Max–plus algebra and its application to railway systems”, G. J. Olsder, (Tool PETER)

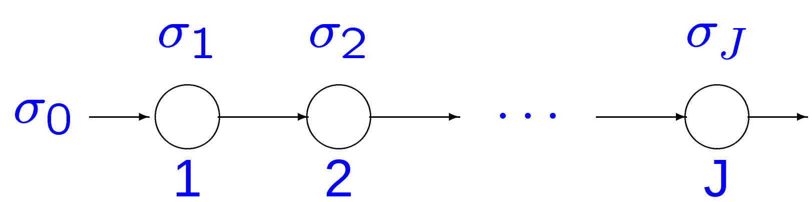
Session TC16, venue WR-11, ”Long-term capacity analysis of tunnels on a railway line”, A. de Kort, B. Heidergott and H. Ayhan.

New! Session TC16, venue WR-11, ”Performance evaluation of train network timetables”, R. Goverde.

Intermezzo: Max Plus Models

A (Max,+)-Linear Queuing System

Consider an open system of J single–server queues in tandem, with infinite buffers. We assume that the system starts empty.



The sequence of departure times then follows

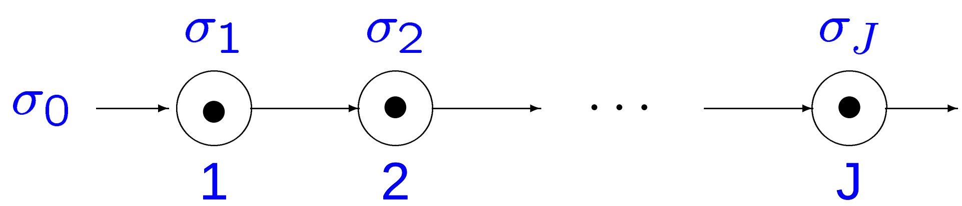
$$x(k+1) = A \otimes x(k) \ ,$$

with

$$A = \begin{bmatrix} \sigma_0 & \epsilon & \epsilon & \dots & \epsilon \\ \sigma_0 \otimes \sigma_1 & \sigma_1 & \epsilon & \dots & \epsilon \\ \sigma_0 \otimes \sigma_1 \otimes \sigma_2 & \sigma_1 \otimes \sigma_2 & \sigma_2 & \dots & \epsilon \\ \vdots & \vdots & \vdots & & \vdots \\ \sigma_0 \otimes \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_J & \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_J & \sigma_2 \otimes \cdots \otimes \sigma_J & \dots & \sigma_J \end{bmatrix} .$$

A (Max,+)-Linear Queuing System

We now consider the open tandem queuing system with one item initially residing at each queue.



The sequence of departure times then follows

$$x(k+1) = A \otimes x(k) \, ,$$

where

$$A = \begin{bmatrix} \sigma_0 & \epsilon & \dots & \epsilon & \epsilon \\ \sigma_0 & \sigma_1 & \epsilon & \dots & \\ & \vdots & & & \\ & \dots & \sigma_{J-2} & \sigma_{J-1} & \epsilon \\ & \dots & \epsilon & \sigma_{J-1} & \sigma_J \end{bmatrix} .$$

(Max,+)-Linear Models

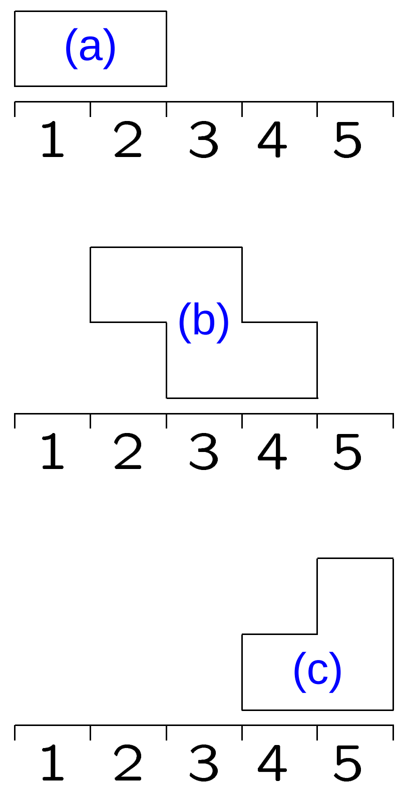
(Max,+) models describe points in time, e.g., when a certain event occurs for the k^{th} time. We have no information about the physical state of the system.

A system is (max,+)–linear if and only if it can be modeled by a FIFO event graph (i.e, a Petri–net such that each place has exactly one up-stream and one downstream transition).

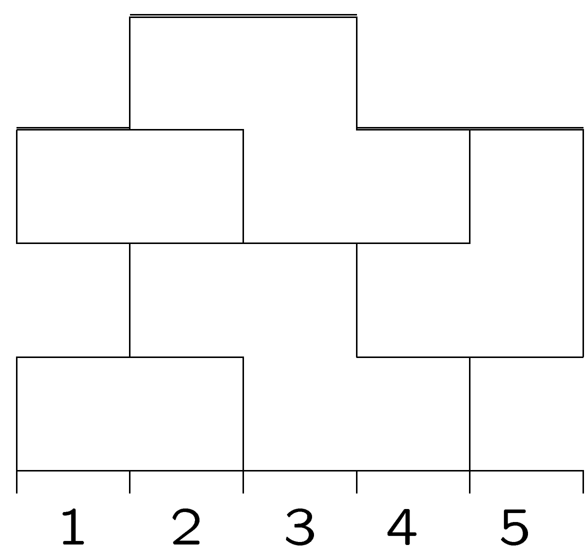
In terms of queuing:
(-) customer classes, (-) overtaking of customers, (-) routing;
(+) synchronization, (+) fork and join, (+) blocking

Max Plus for Stochastic Systems

$J = \{1, 2, \ldots, 5\}$ a set of resources.



The heap $a\,b\,a\,c\,b$:



Heaps of Pieces: The (Max,+) Model

Denote by $x_j(k)$ the height of the pile of pieces over resource j after k blocks.

It can be shown that a piece a can be modeled by a $J \times J$ dimensional matrix $M(a)$.

Piling heaps in a stochastic way leads to a sequence $\{M(k)\}$ of matrices and the height vector follows

$$x(k+1) = M(k) \otimes x(k) , \quad k \geq 0 ,$$

where $x(0)$ denotes the initial contour.

The height vector of the heap $(a\, b\, a\, c\, b)$ is given by:

$$x(5) = x(a\, b\, a\, c\, b) = M(b) \otimes M(c) \otimes M(a) \otimes M(b) \otimes M(a) \otimes x(0) .$$

Types of Limits for Heaps of Pieces

The vector

$$\widehat{x}(k) = \Big(x_2(k) - x_1(k), x_3(k) - x_1(k), \ldots, x_J(k) - x_1(k)\Big)$$

describes the contour of the heap.

First order limit: The growth rate may converge with probability one towards a number

$$\lim_{k \to \infty} \frac{1}{k} x(k) = a \, .$$

Second order limit: The upper contour may converge weakly towards a limiting random variable $\widehat{x}$:

$$\lim_{k \to \infty} \widehat{x}(k) \, .$$

First Order Limits

First Order Ergodic Theory for Stochastic (Max,+)–Linear Systems

Basic assumption: $\{A(k)\}$ is a random sequence such that $A(k)$ has a.s. at least one entry different from ϵ on each row.

We study the sequence

$$\begin{aligned} x(k+1) &= A(k) \otimes x(k)\,, \quad k \in \mathbb{N}\,, \\ x(0) &= x_0\,, \end{aligned}$$

with x_0 finite.

Let

$$||x(k)||_\wedge = \min\{ x_j(k) \}$$

and

$$||x(k)||_\vee = \max\{ x_j(k) \}\,.$$

The First Order Ergodic Theorem

Let $\{A(k)\}$ be an i.i.d. sequence of integrable matrices (with the ”basic assumption” in force).

By Kingman’s subadditive ergodic theorem, finite constants λ^{top} and λ^{bot} exist such that for all finite initial conditions x_0

$$\lim_{k\to\infty} \frac{||x(k)||_\wedge}{k} = \lim_{k\to\infty} E\left[\frac{||x(k)||_\wedge}{k}\right] = \lambda^{bot}$$

and

$$\lim_{k\to\infty} \frac{||x(k)||_\vee}{k} = \lim_{k\to\infty} E\left[\frac{||x(k)||_\vee}{k}\right] = \lambda^{top}$$

with probability one.

λ^{top} is called the maximal or top Lyapunov exponent and λ^{bot} the minimal or bottom Lyapunov exponent.

Refining the First Order Ergodic Theorem

We say that a random matrix A has a fixed support if an element of A is either with probability one finite, or with probability one equal to ϵ.

Fixed support implies that the values of the elements are random but not their position.

Our queuing example has fixed support, the heaps of pieces models fails to have a fixed support.

The First Order Ergodic Theorem (2^{nd} Version)

Let $\{A(k)\}$ be an i.i.d. sequence of matrices such that

- $A(k)$ is integrable,
- $A(k)$ is irreducible (presupposes fixed support),
- any finite element of $A(k)$ is positive and all diagonal elements of $A(k)$ are finite,

then, for $1 \leq j \leq J$,

$$\lim_{k\to\infty} \frac{x_j(k)}{k} = \lim_{k\to\infty} E\left[\frac{x_j(k)}{k}\right] = \lambda \qquad \text{a.s.}$$

$\lambda = \lambda^{\text{top}} = \lambda^{\text{bot}}$ is called the Lyapunov exponent of $\{A(k)\}$.

The First Order Ergodic Theorem: Extensions

The first order ergodic theorem can be extended to reducible matrices.

The ”fixed support” assumption can be relaxed for matrices with discrete state-space via patterns.

The basic definition is: There is a matrix C in the state–space of $\{A(k)\}$ such that (i) C is irreducible, has a unique eigenvector, and (ii) an $N \in \mathbb{N}$ exists with

$$P\Big(A(N) \otimes A(N-1) \otimes \cdots A(1) = C \Big) > 0 \,.$$

Second Order Limits

Waiting Times

Basic equation:

$$x(k+1) = A(k) \otimes x(k) \oplus \tau(k+1) \otimes B(k) \, ,$$

where $\tau(k)$ denotes the time of the k^{th} arrival to the system. We denote by $\sigma_0(k)$ the k^{th} interarrival time to the system, which implies

$$\tau(k) = \sum_{i=1}^{k} \sigma_0(i) \, , \quad k \geq 1 \, ,$$

with $\tau(0) = 0$.

Then, $W_j(k) = x_j(k) - \tau(k)$ denotes the time the k^{th} customer arriving to the system spends in the system until completion of service at server j.

Waiting Times (cont.)

The vector of k^{th} waiting times, denoted by

$$W(k) = (W_1(k), \ldots, W_J(k))$$

follows the recursion

$$W(k+1) = A(k) \otimes C(\sigma_0(k+1)) \otimes W(k) \oplus B(k), \quad k \geq 0,$$

where $C(h)$ denotes a diagonal matrix with $-h$ on the diagonal and ϵ elsewhere.

The Second Order Theorem: Waiting Times

- Let $\{A(k)\}$ be integrable, all finite elements are a.s. non–negative, and the diagonal elements are a.s. non–negative.

- The sequence $\{(A(k), B(k))\}$ is stationary and ergodic, and independent of $\{\tau(k)\}$.

- The maximal Lyapunov exponent of $\{A(k)\}$ is smaller than $E[\sigma_0(k)]$.

Then, $\{W(k)\}$ converges in strong coupling to an unique stationary regime W, with

$$W = \bigoplus_{j \geq 0} C(\tau(-j)) \otimes \bigotimes_{i=1}^{j} A(-i) \otimes B(-(j+1)) \, .$$

Further Topics

Asymptotic Analysis

Stability theory for (max,+)–linear systems for particular types of input distributions.

Ongoing research is on subexponential distributions.

Asymptotics for subexponential networks at IFORS:

Session TB16, venue WR-11, ”Asymptotics of closed queueing networks with subexponential service times”, H. Ayhan, Z. Palmowski and S. Schlegel.

Session TB16, venue WR-11, ”Subexponential asymptotics for stationary open (max,plus) systems”, F. Baccelli, S. Foss and M. Lelarge.

Computational Issues

In contrast to the deterministic setting there exist no efficient algorithm for computing λ, λ^{top}, λ^{bot} or stationary waiting times.

Ongoing research is on computational approaches via Taylor series expansions.

The Taylor series approach at IFORS:

New! Session TB16, venue WR-11, ”Polynomial algorithms for Taylor expansions of max–plus systems”, A. Jean–Marie and M. Heusch.

Session TC16, venue WR-11, ”Tail probability of waiting times in max–plus–linear systems”, H. Ayhan and D.–W. Son.

Session TD16, venue WR-11, ”Numerical evaluation of max–plus–linear systems”, B. Heidergott.

Concluding Remarks

Beyond (Max,+)

- (min,max,+) systems
- MM calculus
- topical mappings
- monotone separable framework
- etc. ...

Upcoming (Max,+) events

- 6^{th} International Workshop on Discrete Event Systems (WODES'02), Zaragoza, Spain, 2–4 October, 2002.

- International Workshop on Max–Plus Algebra, Birmingham, UK, Juli or August, 2003 (in planning)

- IEEE Control Systems Society: First Multidisciplinary International Symposium on Positive Systems: Theory and Applications, Faculty of Engineering, University of Rome ”La Sapienza” Roma, Italy, 28-30 August, 2003.