Splitting contests*

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Abstract

In this paper we investigate how heterogeneous agents choose among contests with different prizes. We show that if the number of agents is sufficiently small, multiple equilibria can arise. These include perfect-sorting (high-ability agents compete for high prize and low-ability agents for low prize), mixed strategies and reversed sorting. We show that total effort always decreases compared to a single contest. However, splitting the contest may increase the effort of low-ability agents.

Keywords: self-selection, Tullock, heterogeneous agents, effort.

1 Introduction

Since Lazear and Rosen (1981) it has been recognized that contests can be very useful in providing incentives. Lazear and Rosen, but also Green and Stokey (1983) and Nalebuff and Stiglitz (1983) analyze internal labor markets as contests. An important theoretical result is that rank-order contests can yield optimal level of effort while, in contrast to piece rates, only relative instead of absolute performance has to be observed to provide correct incentives.[1]

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[1] The empirical evidence in support of this model comes mainly from laboratory experiments (Bull et al., 1987; Harbring and Irlenbusch, 2003; Schotter and Weigelt, 1992) and from sports tournaments (Ehrenberg and Bognanno, 1990; Sunde, 2009). Leuven et al. (2011) argue that due to sorting the external validity of this evidence is limited.

More recent contributions focus on the optimal prize structure of contests (Glazer and Hassin, 1988; Moldovanu and Sela, 2001; see also Clark and Riis, 1998) or on the optimal structure of contests (Moldovanu and Sela, 2006). Moldovanu and Sela (2001) analyze the question how a fixed amount of prize money should be divided between several possibly different prizes in a single contest (a first prize, a second prize, etc.). In their model the agent choosing the highest effort wins the first prize, and agents have private information about their ability (cost of effort). They show that if the cost function of effort is linear or concave then total effort is maximized when there is only one large prize. It might be optimal to split the prize money in more than one prize, but only when the cost function of effort is sufficiently convex. In a similar framework, Moldovanu and Sela (2006) analyze whether it is better to organize one pooled contest, or a series of sub-contests whose winners compete against each other and whose losers are eliminated. The answer to this question now depends on the objective of the organizer (maximize expected total effort or maximize expected highest effort) and again on the curvature of the cost function of effort.[2]

In this paper we analyze the related but different case in which the organizer of a contest has the option to allocate the available prize money over several parallel contests, after which participants have to choose which contest to enter.[3] While Moldovanu and Sela mentioned the interest of this issue already ten years ago, we are not aware of any other study addressing this question.[4] An explanation for this might be that in their framework, where order statistics play an important role, incorporating participants' self-selection into the analysis is not innocuous.

We adopt the model of Tullock (1980), but allow the total prize money to be split (in unequal shares) over two contests. The framework of Tullock is used often in the contest literature. Ryvkin (2011) uses it to study the competition between groups and how heterogeneous agents should be devided among the groups. And also with a Tullock setting, Azmat and Möller (2009) study the competition between organizers of contests. Organizers choose the prize structure to compete for homogeneous agents.

[2]Szymanski (2003) provides a survey of different contest forms in sports, and gives many examples.

[3]In the earlier working paper Leuven et al. (2010) we give a number of illustrating examples of real-life parallel contests.

[4]In their concluding section, Moldovanu and Sela (2001) write that "[a]nother interesting extension would be the study of several parallel contests (with potentially different prize structures), such that agents can choose where to compete." And in the concluding section of Moldovanu and Sela (2006), they state that "[a]nother important avenue for future research is embedding the present analysis in a model of competition among contest designers. ... Since the contest architecture influences the expected payoffs of the participating agents, it is interesting to analyze which agents engage in which contests."

Like Stewart (1999) we allow agents to differ in ability. We also allow the prizes to differ between contests. Heterogeneous agents have to decide either to enter the contest with the high prize or with the low prize. After learning who participates in which contest, all agents decide how much effort to devote to winning the prize in their contest.

Our study relates to the recent literature on competing auctions. Ellison et al. (2004) study the case in which sellers and buyers simultaneously choose between different auction sites. In equilibrium multiple auction sites can coexist. Moldovanu et al. (2008) allow two sellers to first choose location and quantity before buyers decide about their location. Like in Ellison et al. (2004) buyers are identical when choosing their auction site, i.e. they do not know their valuation yet. In their setting there is a movement towards a dominant auction site. Like in Moldovanu et al. (2008) we consider a finite number of agents.[5] Our contest setting differs in two crucial aspects from both auction settings. First, agents are not identical when choosing their location, i.e. they know their type. And second, the existence of different locations is a choice of the single contest organizer rather than the result of competing auctioneers. In our case the contest organizer is maximizing a single objective function rather than various sellers each maximizing their own profit.

Within our relatively simple framework we show that when agents select their contest different equilibria can arise. If the difference in prize money between the contests is sufficiently large, higher ability agents are more likely to sort into the contest with the high prize than low-ability agents. However, if this difference is relatively small there may be multiple equilibria also including equilibria with reversed sorting by ability. Furthermore, we show that splitting contests does not increase total effort devoted by all agents compared to having only a single contest. There are, however, cases when splitting the contest results in more effort from low-ability agents. A principal or organizer who cares about the effort of low-ability agents may thus decide to split the prize over multiple contests.

The outline of the paper is as follows. Section 2 provides the case where there is only a single contest in which all agents participate. In Section 3 we discuss splitting the prize money over two contests. In Section 4 we allow an organizer to assign agents to contests and set the prizes. We also consider alternative objective functions of the organizer in this section. We discuss an alternative costs function of effort in Section 5. Section 6 concludes.

[5] As we will discuss, our contest setting is not really interesting if the number of buyers is very large.

2 A simple contest setting

In this section we consider a simple contest setting which largely resembles Stewart (1999). Consider a setting with two types of agents. There are N_L low-ability agents having high constant marginal costs c_L of effort $e \geq 0$, and N_H high-ability agents with low marginal costs c_H of effort; $c_L > c_H > 0$. All agents participate in a contest, where only the winner gets a prize M. Like Tullock (1980), we define the probability that agent i wins the prize by

$$p_i = \frac{e_i}{\sum_j e_j} \tag{1}$$

This success probability follows from the score function $s_i = \log(e_i) + \epsilon_i$, where ϵ_i follows an extreme type-I distribution, and the winner of the contest is the agent with the highest score. The expected utility of a risk-neutral agent equals

$$u_i = \frac{e_i}{\sum_j e_j} M - c_i e_i \tag{2}$$

Agents choose their effort to maximize expected utility. In equilibrium, all low-ability agents devote the same effort e_L, and all high-ability agents devote effort e_H.

Proposition 1. *In equilibrium the optimal effort of low-ability agents is given by*

$$e_L^* = \begin{cases} \frac{(N_L+N_H-1)(c_L+c_H N_H - c_L N_H)}{(N_L c_L + N_H c_H)^2} M & \text{if} \quad N_H \leq \frac{c_L}{c_L - c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L - c_H} \end{cases} \tag{3}$$

and the optimal effort of high-ability agents equals

$$e_H^* = \begin{cases} \frac{(N_L+N_H-1)(c_H+c_L N_L - c_H N_L)}{(N_L c_L + N_H c_H)^2} M & \text{if} \quad N_H \leq \frac{c_L}{c_L - c_H} \\ \frac{(N_H-1)c_H}{c_H^2 N_H^2} M & \text{if} \quad N_H > \frac{c_L}{c_L - c_H} \end{cases} \tag{4}$$

which implies that if $N_L \geq 1$ and $N_H \geq 1$, then $e_L^* < e_H^*$.

Proof. See Appendix A. □

The condition $N_H \leq \frac{c_L}{c_L - c_H}$ for positive effort of low-ability agents ensures that $c_L + c_H N_H - c_L N_H \geq 0$, so that for both types of agents effort will never be negative. Note that this condition does not depend on the size of the prize M. Low-ability agents only participate in the contest if the number of high-ability agents is limited. The upper bound for the number of high-ability agents depends on the relative difference in marginal costs of effort between low-ability and high-ability agents. Furthermore, if there is only one high-ability agent in the contest, low-ability agents will always devote positive effort regardless of the relative difference in costs.

The organizer of the contest might be interested in the total effort devoted by all agents, which is given by

$$N_L e_L^* + N_H e_H^* = \begin{cases} \frac{N_L+N_H-1}{N_L c_L + N_H c_H} M & \text{if} \quad N_H \leq \frac{c_L}{c_L-c_H} \\ \frac{N_H-1}{c_H N_H} M & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases} \tag{5}$$

Total effort linearly increases with the prize M. Furthermore, total effort is increasing in the number of low-ability agents N_L as long as $N_H \leq \frac{c_L}{c_L-c_H}$, and otherwise total effort is unaffected by N_L. Increasing the number of high-ability agents N_H always increases total effort. This implies that adding additional agents to the contest can never reduce total effort devoted by all agents. In Section 4 we also consider an alternative objective function for the organizer of the contest, i.e. maximizing the minimum effort among all agents. When participating in the same contest, low-ability agents always devote less effort than high-ability agents.

Using the expressions for total effort, we can show that expected utility of a high-ability agent is

$$u_H = \begin{cases} \frac{(c_H+c_L N_L - c_H N_L)^2}{(N_L c_L + N_H c_H)^2} M & \text{if} \quad N_H \leq \frac{c_L}{c_L-c_H} \\ \frac{1}{N_H^2} M & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases} \tag{6}$$

and of a low-ability agent

$$u_L = \begin{cases} \frac{(c_L+c_H N_H - c_L N_H)^2}{(N_L c_L + N_H c_H)^2} M & \text{if} \quad N_H \leq \frac{c_L}{c_L-c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L-c_H} \end{cases} \tag{7}$$

Of course, the expected utility of a high-ability agent is higher than the expected utility of a low-ability agent provided that there are multiple agents, i.e. $N_H + N_L > 1$.

3 Splitting the contest

Next, we split the prize into two prizes αM and $(1-\alpha)M$ with $0.5 \leq \alpha \leq 1$. The prizes can be won in two different contests, and each agent is allowed to participate in only one of the two contests. The game is such that first agents choose whether they want to participate in the contest with the high prize αM or the low prize $(1-\alpha)M$. After agents have chosen their contest, they observe the contest choices of all other agents, and determine their level of effort. Below, we provide a characterization of the equilibria for different values of α. When choosing the contest we impose symmetry with respect to the own group. Agents with the same cost function of effort always follow the same strategy.

3.1 All high-prize equilibrium

The first possible equilibrium we consider is the one where all agents (of both types) choose to participate in the contest with the high prize αM. The consequence is that a deviating agent is the only participant in the low-prize contest. With minimum effort this agent wins the prize, and (expected) utility of the deviating agent is $(1-\alpha)M$. Recall from the previous section that if a high-ability agent and a low-ability agent participate in the same contest, then the high-ability agent has a higher expected utility. Therefore, if a low-ability agent does not deviate from participating in the high-prize contest, the high-ability agent will not deviate either.

Proposition 2. *All agents choose the high-prize contest if the following conditions are satisfied:*

$$N_H \leq \frac{c_L}{c_L - c_H} \tag{8}$$

and

$$\alpha \geq \frac{\left(N_L c_L + N_H c_H\right)^2}{\left(N_L c_L + N_H c_H\right)^2 + (c_L + c_H N_H - c_L N_H)^2} \tag{9}$$

Proof. See Appendix B. □

Note that both conditions do not depend on the total prize M. If the number of high-ability agents is sufficiently small such that $N_H \leq \frac{c_L}{c_L - c_H}$, then the right-hand side of the second condition is less than or equal to 1. Furthermore, the right-hand side of the second condition is increasing in both N_H and N_L implying that if the number of agents increases, the fraction of the total prize going to the high-prize contest should become higher to ensure that all agents choose this contest. If the conditions above are satisfied, this equilibrium is unique.

It may be obvious that setting α such that all agents choose to participate in the high-prize contest cannot increase total effort devoted by all agents. Recall from the previous section that total effort of all agents in a single contest is linearly increasing in the contest's prize. So if all agents participate in a contest with a prize αM rather than M as would be in a single contest, then all agents reduce their effort proportionally. The same holds for the expected utility of the agents, which is also a linear function of the prize money in the contest.

To illustrate when this equilibrium exists we consider a numerical example. In this example we normalize the marginal costs of effort of the low-ability agents to one, i.e. $c_L = 1$. We consider two specific cases, first $N_H = N_L = 2$, and second $N_H = N_L = 4$. So one case with relatively few agents and one case with more agents. Figure 1 shows for which $0 < c_H < 1$ and $0.5 \leq \alpha \leq 1$ the all high-prize equilibrium arises. This equilibrium is only possible if α is relatively close to 1, which implies

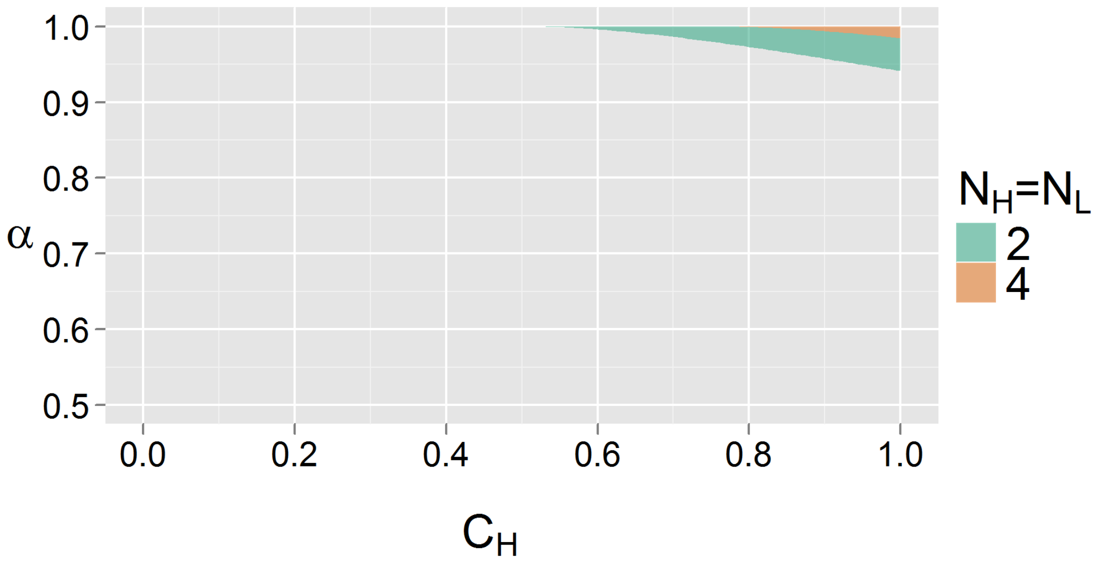

Figure 1. All-high-prize equilibrium

that almost all available prize money should be assigned to the high-prize contest. Furthermore, the difference in marginal costs of effort between the high-ability and low-ability agents should not be too large, which is particularly true if the number of agents increases. In general the area at which in equilibrium all agents choose the high-prize contest becomes smaller as the number of agents increases.

3.2 Perfect-sorting equilibrium

Next, we consider a second equilibrium in which the low-ability agents choose to participate in the contest with the low prize $(1-\alpha)M$ and the high-ability agents participate in the contest with the high prize αM. With this type of perfect sorting, the optimal effort of the high-ability agents becomes

$$e_H^* = \frac{(N_H - 1)c_H}{c_H^2 N_H^2}\alpha M \tag{10}$$

and of the low-ability agents

$$e_L^* = \frac{(N_L - 1)c_L}{c_L^2 N_L^2}(1-\alpha)M \tag{11}$$

There are two equilibrium conditions. First, high-ability agents should prefer the high-prize contest over competing in the low-prize contest with all the low-ability agents. And second, the low-ability agents should not want to deviate to the high-prize contest competing with all the high-ability agents.

Proposition 3. *Under the following conditions perfect sorting is an equilibrium: If*

$N_H \leq \frac{c_L}{c_L - c_H}$, then

$$\frac{N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2}{(c_H + c_L N_L)^2 + N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2} \leq \alpha \leq \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2} \tag{12}$$

Or if $N_H > \frac{c_L}{c_L - c_H}$, then

$$\frac{N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2}{(c_H + c_L N_L)^2 + N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2} \leq \alpha \leq 1 \tag{13}$$

Proof. See Appendix B. □

The lower-bound restriction is strictly smaller than 1. And both the lower bound and the upper bound increase as the number of high-ability agents N_H increases. In the first inequality, the upper bound decreases in the number of low-ability agents N_L, and this is also the case for the lower bound in both inequalities. If c_H becomes larger the lower bound decreases, as well as the upper bound. The intuition behind this is that if the marginal costs of effort of the high-ability increase, they become more similar to the low-ability agents, and, therefore, the prize money should be divided more equally among both contests to have perfect sorting as an equilibrium.

Using the optimal effort of the high-ability and low-ability agents shown in equations (10) and (11), we can show that in the perfect-sorting equilibrium the total effort of all agents is given by

$$N_L e_L^* + N_H e_H^* = \frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M$$

Proposition 4. *In the perfect sorting equilibrium, total effort of all agents is lower than total effort of all agents in a single contest.*

Proof. See Appendix B. □

A contest organizer who is interested in maximizing total effort should avoid that low-ability agents prefer to participate in the other contest than the high-ability agents. We return to the objective of the contest organizer in Section 4 when we also consider alternative objective functions.

Let us return to the numerical example discussed in the previous subsection. Figure 2 shows the areas where perfect sorting is an equilibrium. The figure indicates

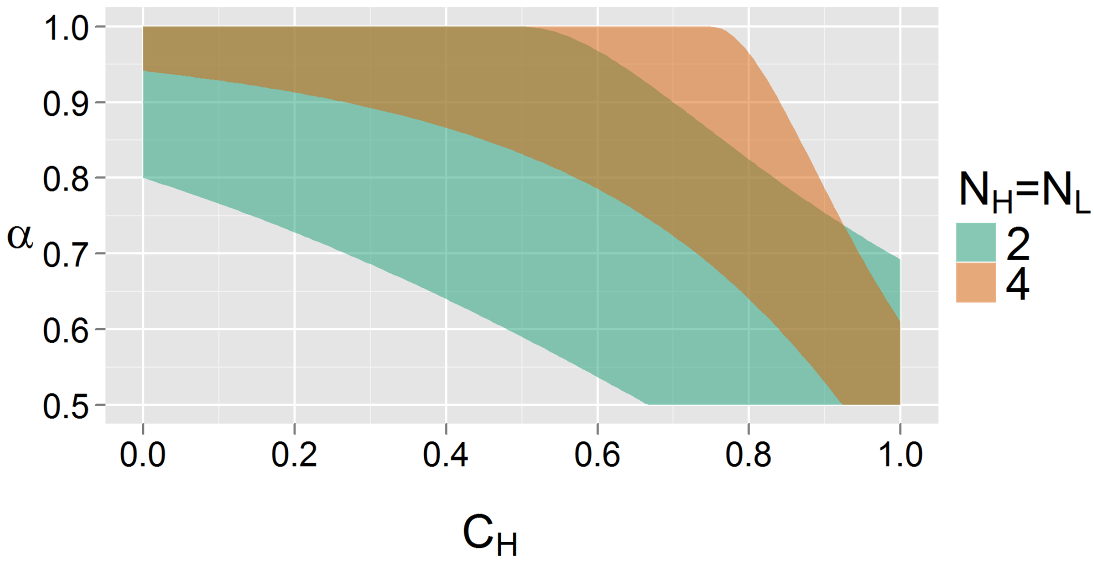

Figure 2. Perfect-sorting equilibrium

that for perfect sorting to be an equilibrium it is either required that the difference in marginal costs of effort between the high-ability and low-ability agents is relatively large (c_H is low), or (if both groups have relatively similar marginal costs of effort) the smaller prize should not be too large (α too small). If the difference in marginal costs of effort is large and the low prize is also substantial, perfect sorting is not an equilibrium. In that case it becomes beneficial for the high-ability agents to enter the low-prize contest, and there is a mixing equilibrium (which we will discuss below). Increasing both the number of high-ability and low-ability agents causes the lower bound to increase. The effect on the upper bound is not monotonic. Recall that the number of high-ability agents and the number of low-ability agents have an opposite effect on the direction in which both bounds move.

Non-uniqueness of the perfect-sorting equilibrium Unlike the equilibrium where all agents sort into the high-prize contest, the perfect-sorting equilibrium is not necessarily unique. When the number of high-ability agents is low, reverse sorting might also be an equilibrium. In that case, all low-ability agents are in the high-prize contest, and the high-ability agents are in the low-prize contest. A high-ability agent prefers to participate in the low-prize contest with only a small number of high-ability agents than to enter the high-prize contest with more low-ability agents.

Proposition 5. *The reverse-sorting equilibrium can arise if the following conditions are satisfied. First,* $N_H \leq \frac{c_H + c_L N_L}{c_H + c_L N_L - c_H N_L}$ *and* $N_L \geq \frac{c_L + c_H N_H}{c_L + c_H N_H - c_L N_H}$. *Second, if* $N_H \leq \frac{c_L}{c_L - c_H}$, *then*

$$\frac{N_L^2 (c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2} \leq \alpha \leq \frac{(c_H + c_L N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 \left(c_H + c_L N_L - c_H N_L\right)^2}$$

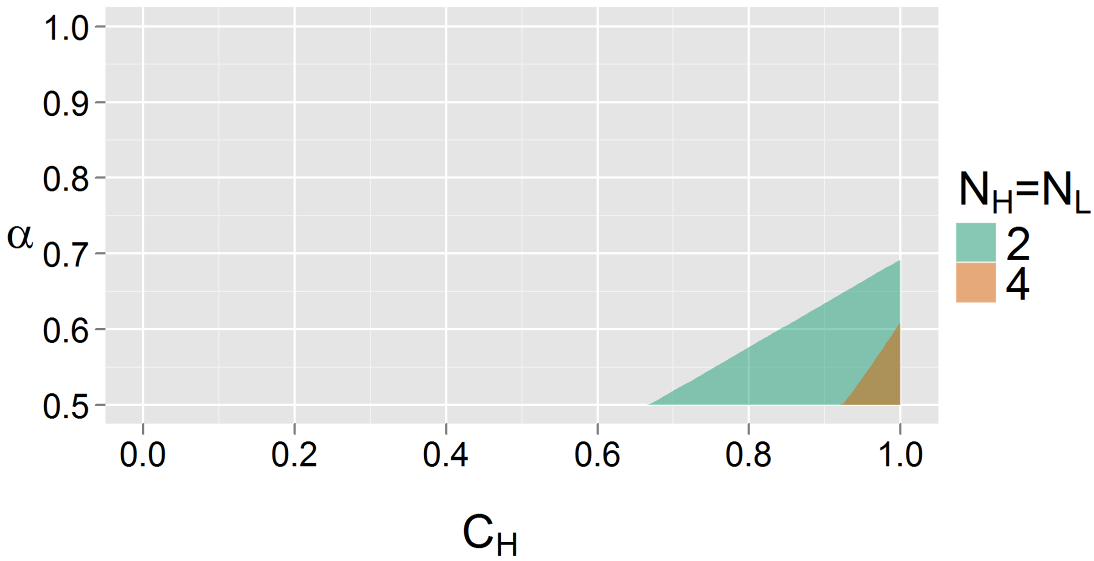

Figure 3. Reverse-sorting equilibrium

Otherwise

$$0.5 \leq \alpha \leq \frac{(c_H + c_L N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 \left(c_H + c_L N_L - c_H N_L\right)^2}$$

Proof. See Appendix B. □

If the number of high-ability agents N_H is large, the upper bound is below 0.5 implying that the reverse-sorting equilibrium is non-existent. However, if N_H is sufficiently small, the upper bound increases in the number of low-ability agents N_L. For the lower-bound condition to exceed 0.5 the number of low-ability agents should be sufficiently large, i.e. $N_L > \frac{c_L + c_H N_H}{c_L + c_H N_H - c_L N_H}$.

Like in the perfect-sorting equilibrium, we can also show for the reverse-sorting equilibrium that total effort of all agents is always lower than in case of a single contest. The proof is the same as the proof of Proposition 4.

Let us again consider the numerical example discussed earlier. Figure 3 shows for which combinations of c_H and α the reverse-sorting equilibrium exists. It is clear that c_H should be relatively close to 1, implying that the marginal costs of effort of the high-ability agents are relatively close to the marginal costs of effort of the low-ability agents. If the difference becomes too large, reverse sorting can no longer be an equilibrium. Furthermore, a substantial part of the total prize should be assigned to the low-prize contest. As already mentioned above the area for which reverse sorting is an equilibrium shrinks rapidly if the number of (high-ability) agents increases. By comparing the area in which reverse sorting is an equilibrium with the area in which perfect sorting is an equilibrium (displayed in Figure 2), one can see that these areas overlap. This means that both equilibria are not necessarily unique.

3.3 Mixed-strategy equilibria

The numerical example discussed above shows that for some parameter values, there are multiple pure-strategy equilibria. Also for some parameter values there are no pure-strategy equilibria. In this subsection, we consider mixed-strategy equilibria. There are a number of mixed-strategy equilibria, which we discuss below. It might be that both types of agents follow a mixed strategy or that one of the types has a pure strategy while the other type has a mixed strategy.

First, consider the possible equilibrium in which both the high-ability agents and the low-ability agents follow a mixed strategy. If α equals 0.5, the total prize is divided equally over both contests. So both contests are the same, and agents are indifferent in which contest they participate. A possible equilibrium is that both a high-ability agent and a low-ability agent choose to enter each of the two contests with probability 0.5. Because both contests are ex-ante the same, none of the agents can improve their outcome by deviating from this mixing strategy.

There is not only a mixed-strategy equilibrium for α being exactly 0.5, but often also for higher values of α. Determining for a given value of the parameters the highest value for α for which it is an equilibrium that both agents have a mixed strategy is more complicated. Therefore, we use the numerical example discussed before and show in Figure 4 for which parameter values it is an equilibrium that both high and low-ability agents mix. The figure shows that for low values of c_H a mixed-strategy equilibrium is possible for higher values of α if there are more agents. For higher values of c_H the figure gets slightly more complicated to interpret. As we will see below, this has to do with other existing mixed-strategy equilibria. In particular, whether or not the low-ability agents may choose a pure strategy to enter the low-prize contest. Furthermore, it should be noted that this mixed-strategy equilibrium partly overlaps with the perfect-sorting equilibrium and the reverse-sorting equilibrium. The intuition is that for values for c_H relatively close to one, α close to 0.5, and two high-ability and two low-ability agents it is an equilibrium if there are (in expectation) in each contests two agents. In that case no agent would want to change because it would imply competing against two agents rather than one.

In a mixed-strategy equilibrium it is difficult to evaluate the (expected) total effort devoted by all agents. Agents only decide about the effort after they learn which agents compete in which contest. Since all agents follow a mixed strategy, the composition of the contests is stochastic and so is total effort. We will show in Section 4 that total effort will always be lower in mixed-strategy equilibria than when only having a single contest (see Proposition 6).

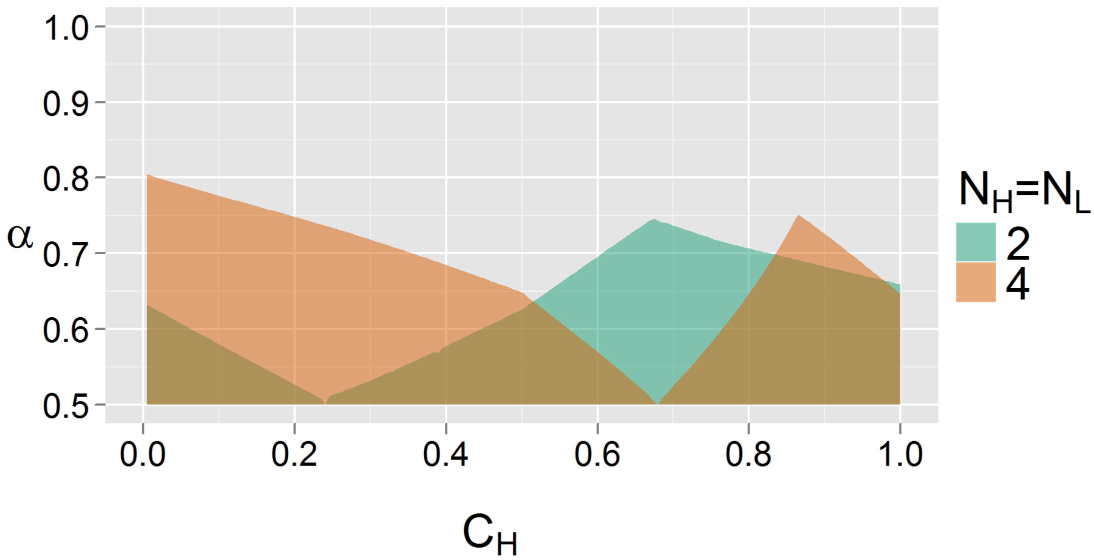

Figure 4. Mixed-strategy equilibrium – where both types mix

From the different figures shown so far, it is clear that there are parameter values for which we have not yet shown any equilibrium. There are two remaining possible mixed-strategy equilibria. The first is the equilibrium in which all high-ability agents decide to participate in the high-prize contest, while the low-ability agents mix between both contests. Recall from Subsection 3.1 that the all high-prize equilibrium is unique. So a first condition for having a mixed-strategy equilibrium in which only low-ability agents mix is that $\alpha < \frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2}$ if $N_H < \frac{c_L}{c_L - c_H}$. Furthermore, a low-ability agent should prefer to enter the high-prize contest if all other low-ability agents enter the low-prize contest. From Subsection 3.2 we know that this implies $N_H < \frac{c_L}{c_L - c_H}$ and $\alpha > \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2}$. This second mixed-strategy equilibrium is thus possible if $N_H < \frac{c_L}{c_L - c_H}$ and

$$\frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2} < \alpha < \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2}$$

This is exactly the area between the all high-prize equilibrium displayed in Figure 1 and the perfect-sorting equilibrium shown in Figure 2.

The final mixed-strategy equilibrium is one where all low-ability agents sort into the low-prize contest and the high-ability agents mix over both contests. Obviously, this equilibrium requires that a high-ability agent should prefer the low-prize contest if all other high-ability agents enter the high-prize contest. The upper bound for α for this mixed-strategy equilibrium is thus the lower bound of the perfect-sorting

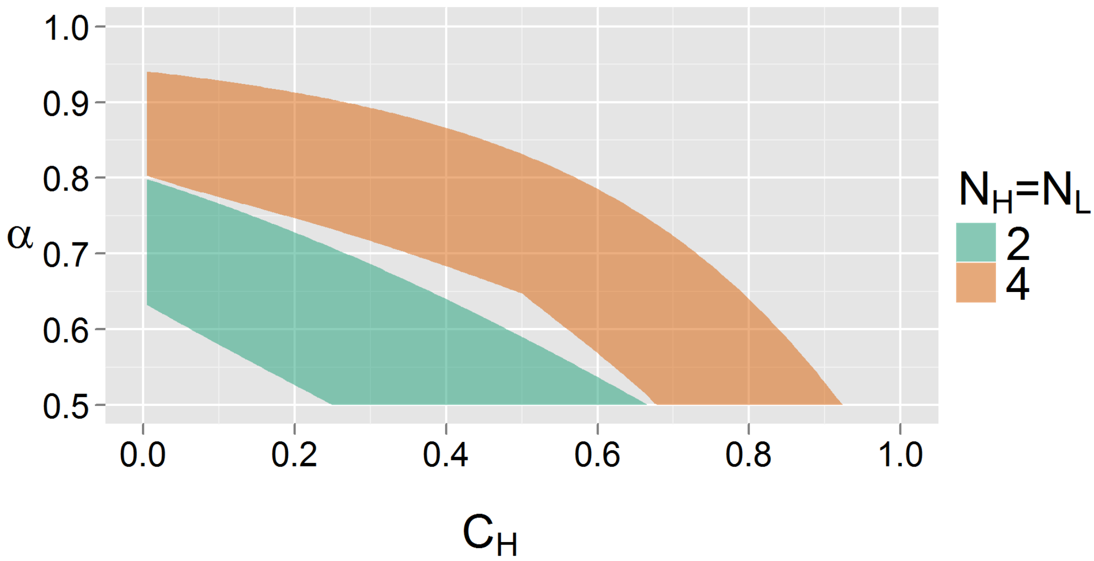

Figure 5. Mixed-strategy equilibrium – where only high-ability types mix

equilibrium, and is given by

$$\alpha < \frac{N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2}{(c_H + c_L N_L)^2 + N_H^2\left(c_H + c_L N_L - c_H N_L\right)^2}$$

Determining the lower bound of this mixed-strategy equilibrium is more complicated and should again be done numerically.

In Figure 5 we again consider the example and show for which values of c_L and α this equilibrium arises. For a setting with only few agents, the value of c_H should be sufficiently small and furthermore α should not be too high. If this is not the case, there will be a perfect selection equilibrium. Furthermore, if both c_H and α are very low, not only the high-ability agents but also the low-ability agents follow a mixed strategy. As can be seen from the figure the area indicating the equilibrium shift substantially if there are more agents in the contest. The latter implies that for more parameter values the equilibrium in unique in which both types of agents use a mixed strategy. Indeed, if there are many agents of both types, the high-ability agents follow a mixed strategy and enter both the low-prize and the high-prize contest. If the number of high-ability agents is sufficiently large, then even low-ability agents in the low-prize contest will not devote any effort. In the limit only the behavior of the high-ability agents is relevant and they mix over both contests.

4 Contest organizer

So far, we have mainly considered the behavior of agents and we have largely ignored the organizer of the contest. We implicitly assumed that the organizer's objective is to optimize total effort of all agents, although we briefly addressed maximizing the minimum effort among all agents as alternative objective function.[6] Below we consider both objective functions. Furthermore, we assume that there is an organizer, who can not only decide about the value of α, but can also assign the agents to the contests.

For illustration we start by considering the case where $N_H = N_L = 2$. Obviously, the contest organizer should make sure that all agents are actually competing, implying that each agent should have at least one competitor. An agent who does not have any competitor earns a prize without providing any effort. Given both objectives of the organizer, this can never be optimal. This leaves us with four possible assignments:

- All four agents are assigned to a single contest in which the full prize money (M) can be won (the pooled contest).
- One high-ability agent and one low-ability agent are assigned to the high-prize contest and one high-ability agent and one low-ability agent are assigned to the low-prize contest (two mixed contests).
- The two high-ability agents are assigned to the high-prize contest and the two low-prize agents are assigned to the low-prize contest (perfect sorting).
- The two high-ability agents are assigned to the low-prize contest and the two low-prize agents are assigned to the high-prize contest (reverse sorting).

We use the results derived earlier to report in Table 1 the maximum total effort and the maximum minimum effort that can be achieved under each of these assignments. Above the equation-signs we report the value of α for which these outcomes are attainable.

It is easy to check that the maximum total effort in the pooled contest is always as least as large as the maximum total effort in case of a split contest. Consider in particular the case of perfect sorting. The best the contest organizer can do in that case is to set $\alpha = 1$. But if all the prize money goes to the high-prize contest in which the high-ability agents participate, it is better to re-assign the low-ability agents also to the high-prize contest. As we have shown in Section 2 increasing the number of agents in a contest will never decrease the total effort in this contest.

It is also easy to check that minimum effort under reverse sorting exceeds minimum effort under perfect sorting and under mixed sorting.[7] Minimum effort under reverse

[6] A contest organizer differs from social planner, since the latter should also include the costs of effort in the objective function.

[7] In both cases this holds because $c_L > c_H$.

Table 1. Outcomes of different assignments by contest organizer

Players	Prize	Maximum total effort	Maximum minimum effort
$\{2H + 2L$	$M \}$	$\frac{3}{2c_L+2c_H}M$ if $c_L \le 2c_H$ $\frac{1}{2c_H}M$ if $c_L > 2c_H$	$\frac{3(2c_H-c_L)}{(2c_L+2c_H)^2}M$ if $c_L \le 2c_H$ 0 if $c_L > 2c_H$
$1H + 1L$ $1H + 1L$	αM $(1-\alpha)M$	$\frac{1}{c_L+c_H}\alpha M + \frac{1}{c_L+c_H}(1-\alpha)M = \frac{1}{c_L+c_H}M$	$\frac{c_H}{(c_L+c_H)^2}(1-\alpha)M \overset{\alpha=0.5}{=} \frac{c_H}{4(c_L+c_H)^2}M$
$2H$ $2L$	αM $(1-\alpha)M$	$\frac{1}{2c_H}\alpha M + \frac{1}{2c_L}(1-\alpha)M \overset{\alpha=1}{=} \frac{1}{2c_H}M$	$\frac{1}{4c_L}(1-\alpha)M \overset{\alpha=0.5}{=} \frac{1}{8c_L}M$
$2H$ $2L$	$(1-\alpha)M$ αM	$\frac{1}{2c_L}\alpha M + \frac{1}{2c_H}(1-\alpha)M \overset{\alpha=0.5}{=} (\frac{1}{4c_L} + \frac{1}{4c_H})M$	$\min(\frac{1}{4c_L}\alpha M, \frac{1}{4c_H}(1-\alpha)M) \overset{\alpha=\frac{c_L}{c_H+c_L}}{=} \frac{1}{4(c_L+c_H)}M$

sorting is, however, not necessarily larger than minimum effort in the pooled contest. This is only true if $c_H < \frac{4}{5}c_L$: the marginal cost of effort of the low-ability agents has to be sufficiently above the marginal cost of effort of the high-ability agents. Otherwise, it is better not to split the prize money and to run the pooled contest.[8] Whether the pooled contest induces higher or lower minimum effort than the mixed contests or the perfect-sorting contests, also depends on the values of c_L and c_H. Mixed contests dominate the pooled contest as long as $c_H < \frac{3}{5}c_L$. Perfect-sorting contests dominate the pooled contest if $10c_Hc_L - c_H^2 < 7c_L^2$, which holds if c_H is sufficiently small compared to c_L.

We can generalize the results regarding maximum total effort to other values of N_H and N_L. Once the contest organizer has divided all agents over both contest, the optimal strategy would be to assign the full prize to one of the contests, i.e. the contest with the highest "weight", because effort within each contest is linear in the prize. If the full prize has been assigned to a single contest, it is also optimal to assign all agents to this contest, because more competition can never reduce total effort. This argument stresses that splitting a contest can never increase total effort of all agents and can thus never be the optimal strategy of a principal or organizer who has the main objective to optimize total effort. A principal or contest organizer who divided the prize over multiple contests thus puts in their objective function some weight on the effort of the low-ability agents.

Proposition 6. *Total effort of all agents is always higher in a single contest with prize money M than in after splitting the prize money over two contests.*

Proof. See Appendix C. □

5 Convex cost function

Above we assumed that costs lineairly increase in effort. In this section, we briefly consider a convex cost function. In particular, we assume that costs increase quadratic in effort $c_i(e_i) = \frac{1}{2}c_ie_i^2$. Furthermore, for ease of computation we simplify the analysis by imposing that there are only two high-ability and two low-ability agents, $N_H = N_L = 2$. First, consider again a single contest with a prize M.

Proposition 7. *In a single contest, optimal effort of the low-ability agents is a fraction of optimal effort of the high ability agents $e_L^* = f(c_L, c_H)e_H^*$, and the optimal*

[8] Note, however, that this situation will never occur when agents self-select into contests, since reverse sorting is only an equilibrium if the costs of high-ability and low-ability agents are sufficiently equal.

effort of the high-ability agents equals

$$e_H^* = \frac{\sqrt{M}}{\sqrt{c_H}} \frac{\sqrt{2f(c_L, c_H) + 1}}{(2f(c_L, c_H) + 2)} \tag{14}$$

with $f(c_L, c_H) = \frac{(c_L - c_H) + \sqrt{(c_L - c_H)^2 + 16 c_L c_H}}{4c_L}$.

Proof. See Appendix D. □

Using the optimal effort in the proposition, we can show that total effort of all agents equals $2e_L^* + 2e_H^* = \frac{\sqrt{M}}{\sqrt{c_H}} \sqrt{2f(c_L, c_H) + 1}$.

Next, we split the contest into two contests with prize αM and $(1 - \alpha) M$, respectively. There are two relevant cases. First, perfect sorting in which both high-ability agents compete against each other in one contest, and both low-ability agents compete against each other in the other contest. Second, mixed sorting when in each contest one high-ability and one low-ability agent compete against each other. If we would divide the four agents differently over the two contest, there would be one contest without participants or with only a single participant. It may be clear that this will not increase total effort.

Let us start with perfect sorting. Both high-ability agents participate in the contest with prize share α, and the low-ability agents compete against each other in the contest with prize share $(1 - \alpha)$. We restrict α to be between 0 and 1.

Proposition 8. *Under perfect sorting, the optimal effort of the high-ability agents equals*

$$e_H^* = \frac{1}{2} \frac{\sqrt{\alpha M}}{\sqrt{c_H}}$$

and of the low-ability agents

$$e_L^* = \frac{1}{2} \frac{\sqrt{(1 - \alpha) M}}{\sqrt{c_L}}$$

Proof. See Appendix D. □

If under perfect sorting the organizer of the contest wants to maximize total effort, it should be that $\alpha^* = \frac{c_L}{c_H + c_L}$. Because $c_L > c_H$, a larger share of the prize should go to the contest with the high-ability agents. In that case total effort of all agents equals $\frac{\sqrt{c_H + c_L}}{\sqrt{c_H c_L}} \sqrt{M}$. Like in the single contest total effort increases linearly in $\sqrt{M}$. If the organizer of the contest is interested in maximizing minimum effort, then α should be equal to $\frac{c_H}{c_H + c_L}$. So a larger share of the prize should be assigned to the contest with the low-ability agents.

Next, consider the case in which in each contest one high ability and one low ability agent participate.

Proposition 9. *The optimal effort of a high-ability agent in the contest with prize αM equals*

$$e_H^* = \sqrt{\frac{\sqrt{c_L}}{\sqrt{c_H}} \frac{\sqrt{\alpha M}}{\sqrt{c_L} + \sqrt{c_H}}}$$

and the optimal effort of the low-ability agent equals

$$e_L^* = \sqrt{\frac{\sqrt{c_H}}{\sqrt{c_L}} \frac{\sqrt{\alpha M}}{\sqrt{c_L} + \sqrt{c_H}}}$$

Proof. See Appendix D. □

It is optimal to split the prize equally over both contests ($\alpha = \frac{1}{2}$). This division of the prize not only maximizes total effort, but also maximizes minum effort. This in not surprising since the composition of both contests is the same. In that case total effort equals $\frac{\sqrt{2M}}{\sqrt{\sqrt{c_L}\sqrt{c_H}}}$, which again increases linearly in $\sqrt{M}$. Furthermore, if we write $c_H = \beta c_L$ with $0 < \beta < 1$, it is easy to show that optimal total effort in three cases can be written as $g_t\left(\beta\right) \frac{\sqrt{M}}{\sqrt{c_L}}$ for the three different contest settings indexed by t. So, when comparing the ranking between contests in total effort only $\beta = \frac{c_H}{c_L}$ is important. Therefore, we normalize both $M = 1$ and $c_L = 1$. In Figure 6 we show total effort in all three contest setups as function of c_H. It is clear that for all parameter values total effort of all agents if higher in case of perfect sorting than in case of mixed sorting. But more importantly, the single contest yields the highest total effort of all agents for all parameter values. This suggests that the results derived in the previous section under a linear cost function also hold in case with a quadratic cost function. It should, however, be stressed that we have only evaluated a setting with only two high-ability and two low-ability agents. Figure 7 shows the minimum effort among all agents for the value of α that maximizes the minimum effort. The figure shows that perfect sorting always yields a higher minimum effort than mixed sorting. Furthermore, for high values of c_H the single contest has a higher minimum effort than perfect sorting, while the opposite is true for low values of c_H. So if high-ability agents and low-ability agents have a cost function of effort which is relatively close letting all agents participate in a single contest maximizes the minimum effort. However, if the cost function of effort differs much between high-ability and low-ability agents, splitting the contest such that agents only participate against their own type can have a higher minimum effort. A requirement is, however, that the low-ability agents compete for a higher prize than

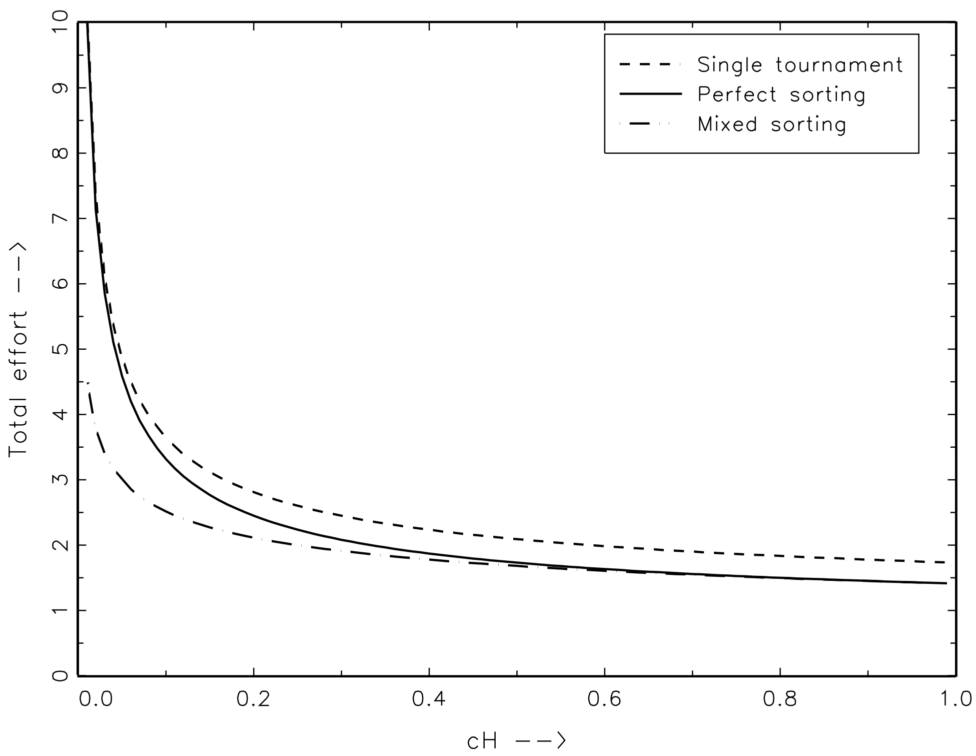

Figure 6. Total effort under quadratic cost function.

the high-ability agents.

6 Conclusions

To examine how heterogeneous agents choose among two contests with different prizes, we analyzed a model in which agents differ in their constant marginal cost of effort and in which an agent's probability to win a contest is equal to the agent's effort relative to the sum of effort of all agents in that contest. We characterize different equilibria and show how these are related to parameter values, especially how the prize money is divided across contests, and the difference in marginal costs between high-ability and low-ability agents.

A pooling equilibrium in which all agents choose for the high-prize contest arises if a large share of the prize money goes to the high-prize contest and if high-ability and low-ability agents are fairly similar in terms of their marginal costs of effort. A perfect sorting equilibrium can arise for less extreme values of these parameters. We also show, however, that this equilibrium is not unique. For some parameter values, both perfect sorting and reverse sorting are equilibria. We also characterize three types of mixed strategy equilibria.

A common finding for all equilibria is that total effort will never exceed the

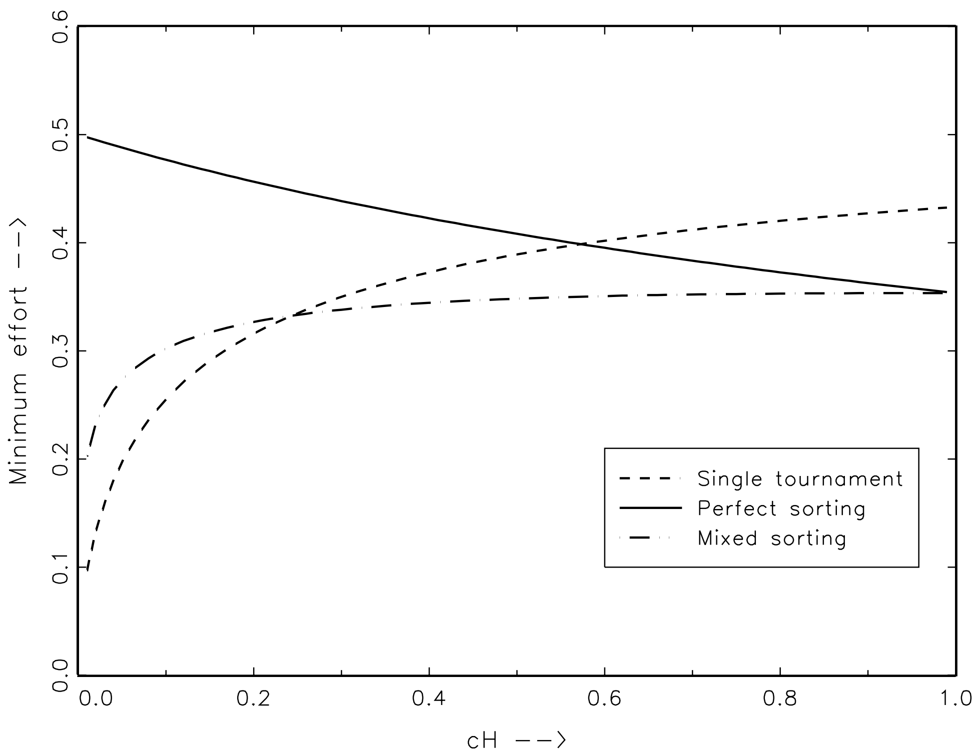

Figure 7. Minimum effort among agents under quadratic cost function.

total effort that is provided when all prize money goes to a single pooled contest. Hence, a principal who wants to maximize total effort should never split a contest in different smaller contests. This result is in line with the conclusions in Moldovanu and Sela (2001) who find that splitting one large prize into various smaller prizes within the same contest will not increase total effort unless agents' cost functions are very convex. We have analyzed a specific case of a convex cost function. Also in this case splitting the prize over different contests does not yield a high total effort. This suggests some robustness of our results, although it should be stressed that we restricted the setting to only two high-ability agents and two low-ability agents to avoid that the model becomes intractable.

We also find that splitting a contest in different smaller contests may have a beneficial impact on the effort level of low-ability agents. This is particularly the case if low-ability and high-ability agents are sufficiently different. A principal or contest organizer who cares about the effort of low-ability agents may thus split the prize over multiple contests.

In a related paper, we report about a field experiment in which first years economics students at the University of Amsterdam had to choose between three contests with different prizes (see Leuven et al., 2011). In one contest the prize was 5000 euros, in another contest the prize was 3000 euros, and in one contest the prize was 1000 euros. Within each contest the best performing student on the final exam of a standard introductory microeconomics course would win the prize. If we use high school math grades as measure of ability, the observed sorting pattern is consistent with students playing mixed strategies. Some high-ability students chose to enter the low-prize contest and some low-ability students entered the high-prize contest. However, on average, more able students are more likely to enter the high-prize contest.

References

Azmat, G. and Möller, M. (2009). Competition among contests. *Rand Journal of Economics*, 40:743–768.

Bull, C., Schotter, A., and Weigelt, K. (1987). Tournaments and piece rates: An experimental study. *Journal of Political Economy*, 95(1):1–33.

Clark, D. and Riis, C. (1998). Competition over more than one prize. *American Economic Review*, 88:276–289.

Ehrenberg, R. G. and Bognanno, M. L. (1990). Do tournaments have incentive effects? *Journal of Political Economy*, 98(6):1307–1324.

Ellison, G., Frudenberg, D., and Möbius, M. (2004). Competing auctions. *Journal of the European Economic Association*, 2(1):30–66.

Glazer, A. and Hassin, R. (1988). Optimal contests. *Economic Inquiry*, 26:133–143.

Green, J. R. and Stokey, N. L. (1983). A comparison of tournaments and contracts. *Journal of Political Economy*, 91:349–364.

Harbring, C. and Irlenbusch, B. (2003). An experimental study on tournament design. *Labour Economics*, 10(4):443–464.

Lazear, E. P. and Rosen, S. (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy*, 89(5):841–864.

Leuven, E., Oosterbeek, H., Sonnemans, J., and Van der Klaauw, B. (2011). Incentives versus sorting in tournaments: Evidence from a field experiment. *Journal of Labor Economics*, 29:637–658.

Leuven, E., Oosterbeek, H., and Van der Klaauw, B. (2010). Splitting tournaments. Discussion Paper 8016, CEPR.

Moldovanu, B. and Sela, A. (2001). The optimal allocation of prizes in contests. *American Economic Review*, 91:542–558.

Moldovanu, B. and Sela, A. (2006). Contest architecture. *Journal of Economic Theory*, 126:70–96.

Moldovanu, B., Sela, A., and Shi, X. (2008). Competing auctions with endogenous quantities. *Journal of Economics Theory*, 141:1–27.

Nalebuff, B. J. and Stiglitz, J. E. (1983). Prizes and incentives: Towards a general theory of compensation and competition. *Bell Journal of Economics*, 14:21–43.

Ryvkin, D. (2011). The optimal sorting of players in contests between groups. *Games and Economic Behavior*, forthcoming.

Schotter, A. and Weigelt, K. (1992). Asymmetric tournaments, equal opportunity laws, and affirmative action: Some experimental results. *Quarterly Journal of Economics*, 107(2):511–539.

Stewart, J. (1999). Adverse selection and pay compression. *Southern Economic Journal*, 65(4):885–899.

Sunde, U. (2009). Heterogeneity and performance in tournaments: a test for incentive effects using professional tennis data. *Applied Economics*, 41(25):3199–3208.

Szymanski, S. (2003). The economic design of sporting contests. *Journal of Economic Literature*, 41:1137–1187.

Tullock, G. (1980). *Towards a theory of the rent-seeking society*, chapter Efficient rent seeking, pages 153–179. Edward Elgar Publishing.

Appendix A

Proof. (Proposition 1). Agent i chooses effort e_i to optimize the utility function in equation (2). The first-order condition is

$$\frac{\partial u_i}{\partial e_i} = \frac{\sum_j e_j - e_i}{\left(\sum_j e_j\right)^2} M - c_i = 0 \tag{15}$$

In equilibrium, all low-ability agents should devote the same optimal effort e_L, and all high-ability agents devote effort e_H. Substitution this in the first-order condition yields

$$\frac{(N_L-1)e_L+N_He_H}{\left(N_Le_L+N_He_H\right)^2}M-c_L=0 \tag{16}$$

and

$$\frac{(N_H-1)e_H+N_Le_L}{\left(N_Le_L+N_He_H\right)^2}M-c_H=0 \tag{17}$$

Combining these equations gives

$$\frac{c_H}{N_Le_L+(N_H-1)e_H}=\frac{M}{\left(N_Le_L+N_He_H\right)^2}=\frac{c_L}{(N_L-1)e_L+N_He_H} \tag{18}$$

which implies

$$e_L=\frac{c_HN_H-c_LN_H+c_L}{c_LN_L-c_HN_L+c_H}e_H \tag{19}$$

Because $c_L>c_H$, the denominator on the right-hand side is always positive. This does not hold for the numerator. Only if $c_HN_H-c_LN_H+c_L>0$, low-income agents devote positive effort. Furthermore, the numerator is decreasing in N_H and the denominator increasing in N_L. Therefore, $e_L\leq\frac{c_H}{c_L}e_H<e_H$.

If $c_HN_H-c_LN_H+c_L\leq 0$, then $e_L^*=0$. In that case we obtain e_H^* from the first-order condition in equation (17). If $e_L=0$, the first-order condition simplifies to $\frac{N_He_H}{(N_He_H)^2}M=c_H$, so $e_H^*=\frac{M}{c_HN_H}$.

If $c_HN_H-c_LN_H+c_L>0$, we should substitute equation (19) in the first-order condition in equation (17) which gives

$$\frac{N_L\left(\frac{c_HN_H-c_LN_H+c_L}{c_LN_L-c_HN_L+c_H}\right)e_H+\left(N_H-1\right)e_H}{\left(N_L\left(\frac{c_HN_H-c_LN_H+c_L}{c_LN_L-c_HN_L+c_H}\right)e_H+N_He_H\right)^2}M-c_H=0 \tag{20}$$

Multiplying the denominator and the numerator with $\left(c_LN_L-c_HN_L+c_H\right)^2$ and collecting terms gives

$$\frac{\left(c_HN_H+c_HN_L-c_H\right)\left(c_LN_L-c_HN_L+c_H\right)}{\left(c_LN_L+c_HN_H\right)^2e_H}M-c_H=0 \tag{21}$$

so

$$e_H^*=\frac{\left(N_H+N_L-1\right)\left(c_LN_L-c_HN_L+c_H\right)}{\left(c_LN_L+c_HN_H\right)^2}M \tag{22}$$

Substitution this in equation (19) gives

$$e_L^*=\frac{\left(N_H+N_L-1\right)\left(c_HN_H-c_LN_H+c_L\right)}{\left(c_LN_L+c_HN_H\right)^2}M \tag{23}$$

□

Note further that $c_L(N_H + N_L - 1) > c_H(N_H + N_L - 1)$ can be rewritten into $c_L N_L - c_H N_L + c_H > c_H N_H - c_L N_H + c_L$, which implies that $e^*_H > e^*_L$.

Appendix B

Proof. (Proposition 2). Recall that when participating in the same contest low-ability agents have a lower expected utility. When all agents participate in the high-prize contest, the expected utility of a low-ability agent is

$$u_L = \begin{cases} \frac{(c_L + c_H N_H - c_L N_H)^2}{(N_L c_L + N_H c_H)^2}\alpha M & \text{if} \quad N_H \leq \frac{c_L}{c_L - c_H} \\ 0 & \text{if} \quad N_H > \frac{c_L}{c_L - c_H} \end{cases} \tag{24}$$

A deviating agent receives $(1-\alpha)M$ without devoting any effort. Agents do not benefit from deviating if two conditions are satisfied. First, $N_H \leq \frac{c_L}{c_L - c_H}$ to ensure that low-ability agents have a positive expected utility in the high-prize contest. And second, low-ability agents choose the high-prize contest if

$$\frac{(c_L + c_H N_H - c_L N_H)^2}{(N_L c_L + N_H c_H)^2}\alpha M \geq (1-\alpha)M \tag{25}$$

Rewriting gives the condition shows

$$\alpha \geq \frac{(N_L c_L + N_H c_H)^2}{(N_L c_L + N_H c_H)^2 + (c_L + c_H N_H - c_L N_H)^2} \tag{26}$$

□

Proof. (Proposition 3). First, consider the high-ability agents. When participating in the contest with the high prize, their expected utility is $\frac{1}{N_H^2}\alpha M$. Switching to the low-prize contest gives an expected utility $\frac{(c_H + c_L N_L - c_H N_L)^2}{(c_H + c_L N_L)^2}(1-\alpha)M$. The condition

$$\frac{1}{N_H^2}\alpha M \geq \frac{(c_H + c_L N_L - c_H N_L)^2}{(c_H + c_L N_L)^2}(1-\alpha)M \tag{27}$$

implies the lower-bound condition

$$\alpha \geq \frac{N_H^2 (c_H + c_L N_L - c_H N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 (c_H + c_L N_L - c_H N_L)^2} \tag{28}$$

Next, consider a low-ability agent. If $N_H > \frac{c_L}{c_L - c_H}$, this agent does not derive utility from participating in the high-prize contest. If $N_H \leq \frac{c_L}{c_L - c_H}$, the expected utility in the low-prize contest is $\frac{1}{N_L^2}(1-\alpha)M$, and in the high-prize contest $\frac{(c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2}\alpha M$.

This gives the condition

$$\frac{1}{N_L^2}(1-\alpha)M \leq \frac{(c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2}\alpha M \qquad \text{if} \qquad N_H \leq \frac{c_L}{c_L - c_H} \tag{29}$$

which provides the upper-bound condition

$$\alpha \leq \frac{(c_L + c_H N_H)^2}{(c_L + c_H N_H)^2 + N_L^2(c_L + c_H N_H - c_L N_H)^2} \qquad \text{if} \qquad N_H \leq \frac{c_L}{c_L - c_H} \tag{30}$$

If the second inequality is not satisfied, the upper-bound restriction simplifies to $\alpha \leq 1$. □

Proof. (Proposition 4). Recall that in the single contest, we distinguish two cases. First, if $N_H \leq \frac{c_L}{c_L - c_H}$, both high-ability and low-ability agents devote positive effort, and total effort is given by $\frac{N_L + N_H - 1}{N_L c_L + N_H c_H}M$. The difference in total effort when splitting the contest compared to the single contest is

$$\frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M - \frac{N_L + N_H - 1}{N_L c_L + N_H c_H}M \tag{31}$$

Because all terms are linear in M, we ignore M and rewrite the expression to

$$c_H N_H^2(c_H(N_L - 1) - c_L N_L) + \alpha c_L N_L^2(c_L(N_H - 1) - c_H N_H) \tag{32}$$

which is always negative. The first term is negative because high-ability agents have lower marginals costs of effort than low-ability agents and, therefore, $c_H(N_L - 1) < c_L N_L$. The second term is only positive if $c_L(N_H - 1) - c_H N_H > 0$, which implies $N_H > \frac{c_L}{c_L - c_H}$. However, if this is satisfied, low-ability agents do not devote any effort, and we should evaluate

$$\frac{N_L - 1}{c_L N_L}(1-\alpha)M + \frac{N_H - 1}{c_H N_H}\alpha M - \frac{N_H - 1}{c_H N_H}M \tag{33}$$

If we again ignore M, we can rewrite this expression to

$$\frac{N_L - 1}{c_L N_L} - \frac{N_H - 1}{c_H N_H} \tag{34}$$

The condition $N_H > \frac{c_L}{c_L - c_H}$ implies $\frac{(N_H - 1)}{c_H N_H} > \frac{1}{c_L}\left(= \frac{N_L}{c_L N_L}\right) > \frac{N_L - 1}{c_L N_L}$. □

Proof. (Proposition 5). This reverse-sorting equilibrium can only arise if the following conditions are satisfied. First, it should not be beneficial for high-ability agents to deviate, which means,

$$\frac{1}{N_H^2}(1-\alpha)M \geq \frac{\left(c_H + c_L N_L - c_H N_L\right)^2}{(c_H + c_L N_L)^2}\alpha M \tag{35}$$

Recall that if there is only a single high-ability agent in a contest, low-ability agents will devote positive effort. The upper-bound condition for the reverse-sorting equilibrium is

$$\alpha \leq \frac{(c_H + c_L N_L)^2}{(c_H + c_L N_L)^2 + N_H^2 \left(c_H + c_L N_L - c_H N_L\right)^2} \tag{36}$$

From this condition it can be seen that if the number of high-ability agents N_H is large ($N_H \leq \frac{c_H + c_L N_L}{c_H + c_L N_L - c_H N_L}$), the upper bound is below 0.5 implying that reverse sorting is not an equilibrium. However, if N_H is small, then the upper bound is above 0.5, which can be verified by considering the case $N_H = 1$. Furthermore, the upper bound increases in the number of low-ability agents N_L. Second, also for a low-ability agent it should not be beneficial to deviate, which implies

$$\frac{1}{N_L^2}\alpha M \geq \frac{\left(c_L + c_H N_H - c_L N_H\right)^2}{(c_L + c_H N_H)^2}(1-\alpha)M \tag{37}$$

The lower bound is thus given by

$$\alpha \geq \frac{N_L^2 (c_L + c_H N_H - c_L N_H)^2}{(c_L + c_H N_H)^2 + N_L^2 (c_L + c_H N_H - c_L N_H)^2} \tag{38}$$

And, of course, $N_H \leq \frac{c_L}{c_L - c_H}$, otherwise the deviating low-ability agent will not devote any effort when entering the low-prize contest with all high-ability agents. For the lower-bound condition to exceed 0.5 the number of low-ability agents should be sufficiently large, i.e. $N_L > \frac{c_L + c_H N_H}{c_L + c_H N_H - c_L N_H}$. □

Appendix C

Proof. (Proposition 6). The contest organizer assigns N_H^h high-ability agents and N_L^h low-ability agents to the contest with the high prize αM. The remaining $N_H^l = N_H - N_H^h$ high-ability agents and $N_L^l = N_L - N_L^h$ low-ability agents participate in the contest with the low prize $(1-\alpha)M$. According to equation (5) the total effort of all agents in the high-prize contest $e(N_H^h, N_L^h, c_H, c_L, \alpha M)$ can be written as $\alpha M \cdot e(N_H^h, N_L^h; c_H, c_L, 1)$. And similarly for the low-prize contest $e(N_H^l, N_L^l, c_H, c_L, (1-\alpha)M) = (1-\alpha)M \cdot e(N_H^l, N_L^l; c_H, c_L, 1)$. This implies that the total effort in the two contests equals

$$\alpha M \cdot e(N_H^h, N_L^h, c_H, c_L, 1) + (1-\alpha) M \cdot e(N_H^l, N_L^l, c_H, c_L, 1) \tag{39}$$

Since this total effort function is linear in α, the optimum is either $\alpha = 1$ if $e(N_H^h, N_L^h, c_H, c_L, 1) \geq e(N_H^l, N_L^l, c_H, c_L, 1)$, or $\alpha = 0$ if $e(N_H^h, N_L^h, c_H, c_L, 1) < e(N_H^l, N_L^l, c_H, c_L, 1)$. So whatever the assignment of both agents over both contests is, it is always optimal to assign all prize money to a single contest. Recall again from equation (5) that total effort is strictly increasing in the number of high-ability agents and non-decreasing in the number of low-ability agents. Therefore, total effort is optimal if all agents are assigned to a contest with all prize money. □

Appendix D

Proof. (Proposition 7). The first-order conditions are

$$\frac{2e_L + e_H}{(2e_L + 2e_H)^2} M - c_H e_H = 0 \tag{40}$$

and

$$\frac{2e_H + e_L}{(2e_L + 2e_H)^2} M - c_L e_L = 0 \tag{41}$$

So

$$\frac{2e_L + e_H}{c_H e_H} = \frac{2e_H + e_L}{c_L e_L} \tag{42}$$

which implies

$$2c_L e_L^2 + (c_L - c_H)\, e_H e_L - 2c_H e_H^2 = 0 \tag{43}$$

Therefore, $e_L^* = f(c_L, c_H) e_H^*$, with $f(c_L, c_H) = \frac{(c_L - c_H) + \sqrt{(c_L - c_H)^2 + 16 c_L c_H}}{4c_L}$. Substituting this in the first-order condition (40) gives

$$\frac{2f(c_L, c_H) + 1}{(2f(c_L, c_H) + 2)^2\, e_H} M - c_H e_H = 0 \tag{44}$$

so

$$e_H^* = \frac{\sqrt{M}}{\sqrt{c_H}} \frac{\sqrt{2f(c_L, c_H) + 1}}{(2f(c_L, c_H) + 2)} \tag{45}$$

□

Proof. (Proposition 8). With only two agents in a contest with prize M', the first-order condition for individual i is

$$\frac{e_j}{\left(e_j + e_i\right)^2} M' - c_i e_i = 0 \tag{46}$$

Because under perfect sorting both individuals in a contest devote the same effort, optimal effort is given by

$$e^* = \frac{1}{2} \frac{\sqrt{M'}}{\sqrt{c}} \tag{47}$$

In the contest with th high-ability agents $M' = \alpha M$ and $c = c_H$, and in the contest with the low-ability agents $M' = (1 - \alpha) M$ and $c = c_L$. $\square$

Proof. (Proposition 9). In the mixed-sorting case, the first-order condition for the high-ability agent in the contest with prize αM is

$$\frac{e_L}{\left(e_L + e_H\right)^2} \alpha M - c_H e_H = 0 \tag{48}$$

and the first-order condition for the low-ability agents is

$$\frac{e_H}{\left(e_L + e_H\right)^2} \alpha M - c_L e_L = 0 \tag{49}$$

Combining these first-order conditions gives

$$e_L = \frac{\sqrt{c_H}}{\sqrt{c_L}} e_H \tag{50}$$

When we substitute this in first-order condition (48), this gives

$$e_H^* = \sqrt{\frac{\sqrt{c_L}}{\sqrt{c_H}} \frac{\sqrt{\alpha M}}{\sqrt{c_L} + \sqrt{c_H}}} \tag{51}$$

and

$$e_L^* = \sqrt{\frac{\sqrt{c_H}}{\sqrt{c_L}} \frac{\sqrt{\alpha M}}{\sqrt{c_L} + \sqrt{c_H}}} \tag{52}$$

$\square$