Advanced Programming in Quantitative Economics

Introduction, structure, and advanced programming techniques

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Outline

Standard deviations

Day 3 - Morning

9.00L Optimization
- Newton-Raphson and quadratic optimisation
- Hessian: Importance and problems
- Loglikelihood and covariance
- MaxBFGS

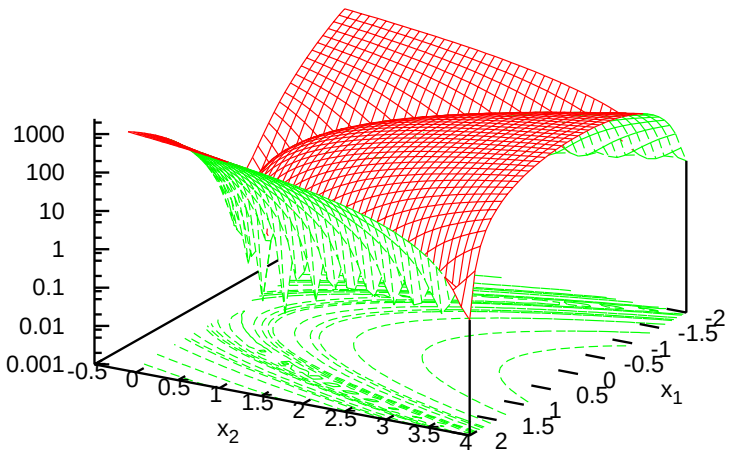
10.30P Estimating a duration model
- Likelihood
- Optimization
- Covariance

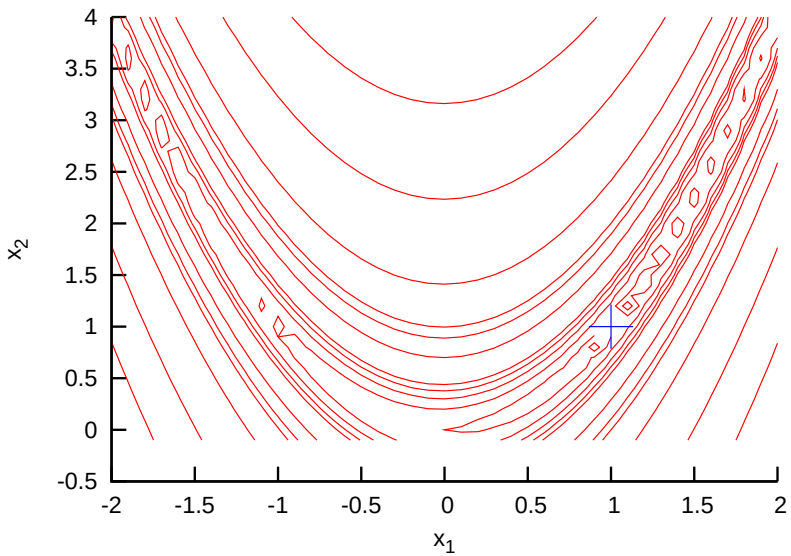
12.00 Lunch

Maximization: Theory

Rosenbrock function:

$$g(x) = 100 * (x_2 - x_1^2)^2 + (1 - x_1)^2$$
$$f(x) = -g(x)$$





Minimum of $g(x)$ at (1, 1) — Steep function — minimum in 'narrow crooked alley'

Wrong approach: Purely random

Suppose:

- Search in area $[-10, 10]^2$
- Throw darts randomly
- Continue until you get precision $\epsilon = 0.25$ from minimum, in both coefficients

How many darts would you need?

$$P(|x_i - x_i^*| < \epsilon) = 2\frac{\epsilon}{20} = .025$$

$$P(|x_i - x_i^*| < \epsilon, i = 1, 2) = P(|x_i - x_i^*| < \epsilon)^2 = 4\frac{\epsilon^2}{20^2} = 0.000625$$

$$E(n) = 1/p = 1600$$

Way too many...

See `rb_darts.ox`

Better approach: Use function characteristics

- Start in some point $x^{(k)}$
- Choose a direction s
- Move distance α in that direction, $x^{(k+1)} = x^{(k)} + \alpha s$
- Increase k, and possibly continue from 1

Direction s: Linked to gradient?
Optimum: Gradient 0, second derivative positive definite?

Ingredients

$f(x) : \Re^n \rightarrow \Re$ Function

$f'(x) = \nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \ldots \frac{\partial f(x)}{\partial x_n}\right]^T \equiv g$ Derivative, gradient

$f''(x) = \nabla^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]_{i,j=1}^{n} \equiv H$ Second derivative, Hessian

If derivatives are continuous, then

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i} \qquad H = H^T$$

Hessian symmetric

Newton-Raphson

- Approach $f(x)$ locally with quadratic function

$$f(x+h) \approx q(h) = f(x) + h^T f'(x) + \frac{1}{2} h^T f''(x) h$$

- Minimise $q(h)$ (instead of $f(x+h)$)

$$q'(h) = f'(x) + f''(x)h = 0 \quad \Leftrightarrow \quad f''(x)h = -f'(x) \quad \text{or} \quad Hh = -g$$

  by solving last expression, $h = -H^{-1}g$

- Choose $x = x + h$, and repeat as necessary

Problems:

- Is H positive definite/invertible, at each step?
- Is step h, of length $||h||$, too big or small?

Problematic Hessian?

Algorithms based on NR need $H^{(k)} = f''(x^{(k)})$. Problematic:

- Taking derivatives is not stable (...)
- Needs many function-evaluations
- H not guaranteed to be positive definite

Problem is in step

$$s_k = -{H^{(k)}}^{-1} g_k$$

Replace ${H^{(k)}}^{-1}$ by some M_k, positive definite by definition?

BFGS/DFP

Broyden, Fletcher, Goldfarb and Shanno (BFGS, choose $\theta = 1$), but also Davison, Fletcher en Powell (DFP, $\theta = 0$) thought of following trick:

1. Start with $k = 0$ and positive definite M_k, e.g. $M = I$
2. Calculate $s_k = -M_k g_k$, with $g_k = f'(x^{(k)})$
3. Find new $x^{(k+1)} = x^{(k)} + h_k$, $h_k = \alpha s_k$
4. Calculate, wtih $q_k = g_{k+1} - g_k$

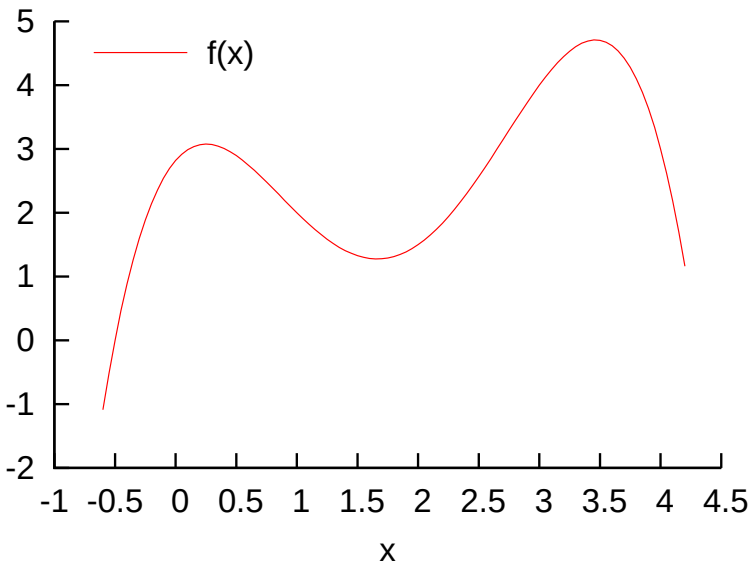
$$M_{k+1} = M_k + \left(1 + \theta \frac{q_k' M_k q_k}{h_k' q_k}\right) \frac{h_k h_k'}{h_k' q_k} - \frac{1-\theta}{q_k' M_k q_k} M_k q_k \; q_k' M_k - \frac{\theta}{h_k' q_k} \left(h_k q_k' M_k + M_k q_k h_k'\right)$$

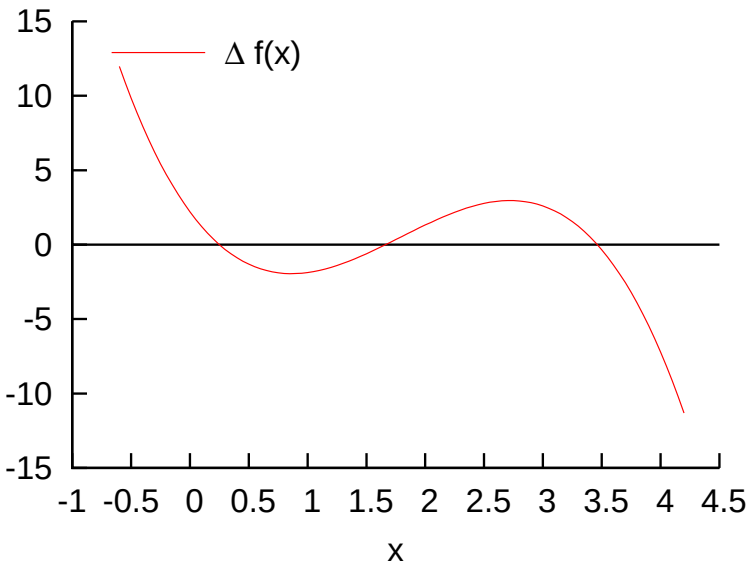
Result:

- No Hessian needed
- Still good convergence
- No problems with negative definite H_k

$\Rightarrow$ `MaxBFGS` in Ox, similar routines in Matlab/Gauss/other.

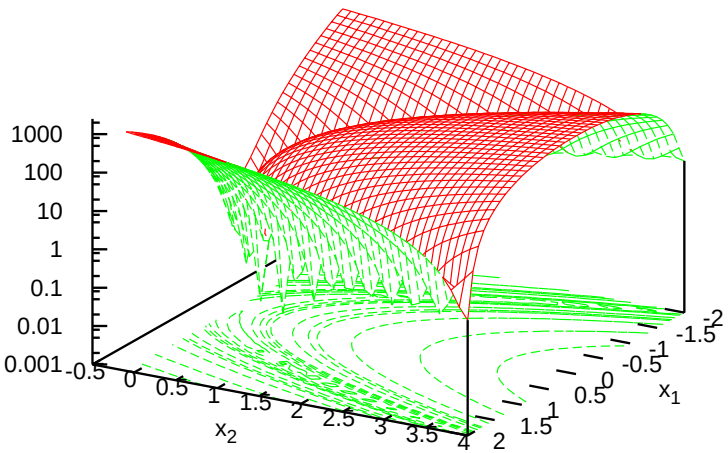
Pitfalls I

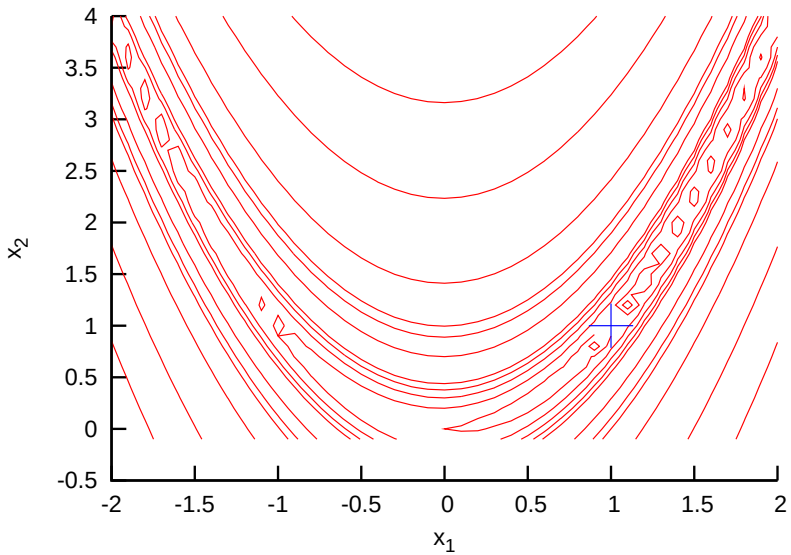




- Local optima: Different starting points? Simulation-based optimisation (simulated annealing, anyone?)

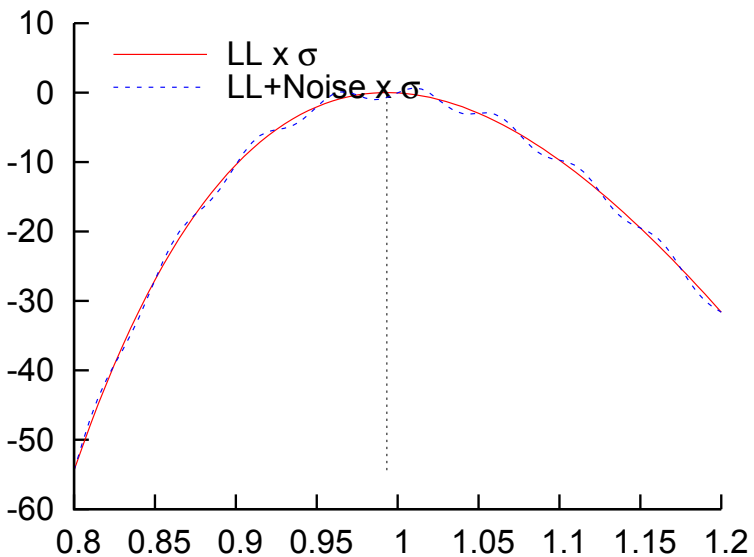
Pitfalls II

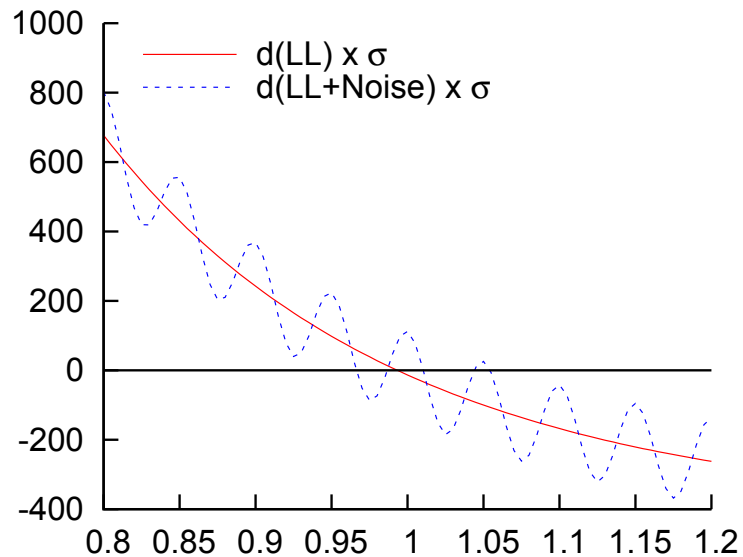




- Flat surface: Different starting points; transformation of parameter ($\theta = \log \sigma$ tends to optimise a lot better...)

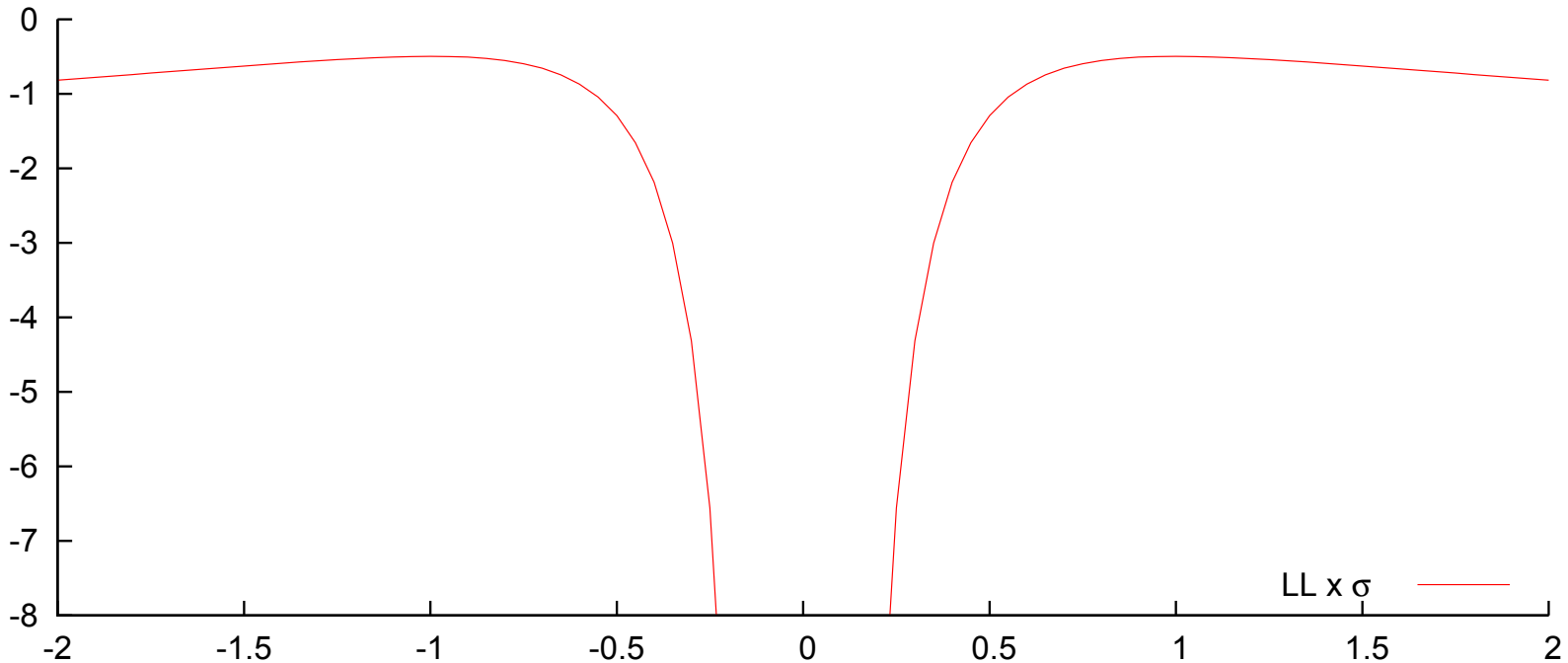
Pitfalls III





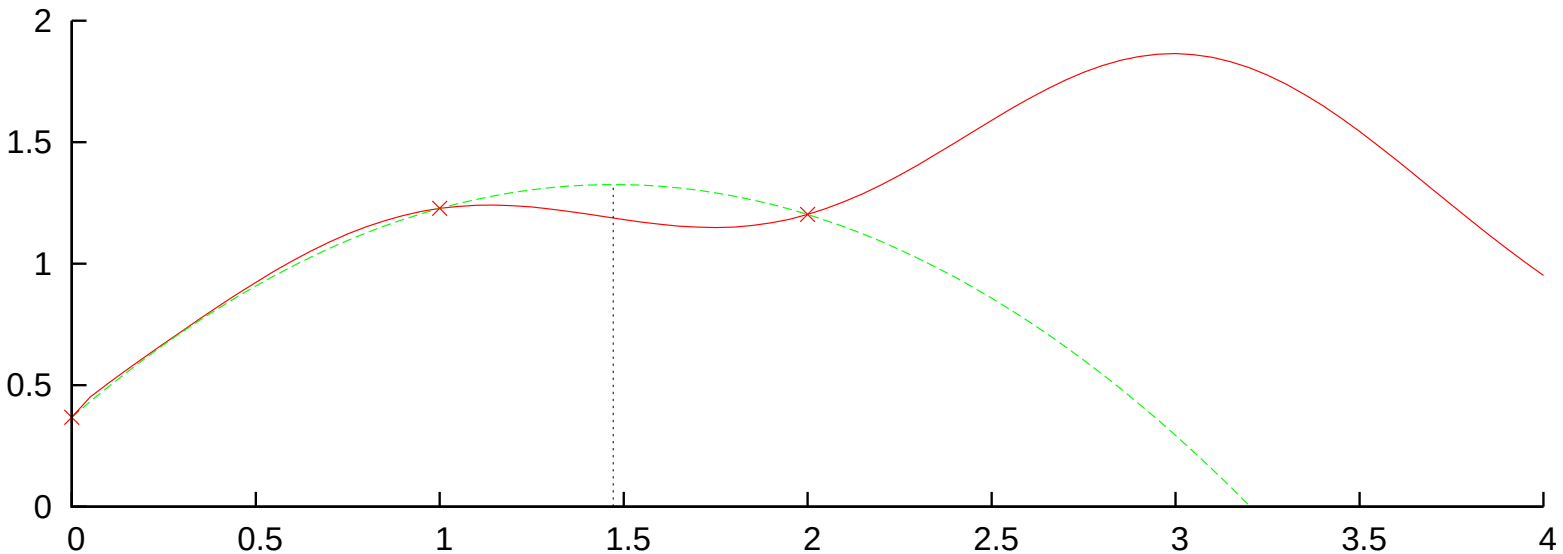
- Noise on derivatives/function: Only smooth functions can be optimised using NR-approach. Use analytical derivatives. Robust programming of target function (optimise AVERAGE LOG likelihood, never likelihood itself).

Pitfalls IV



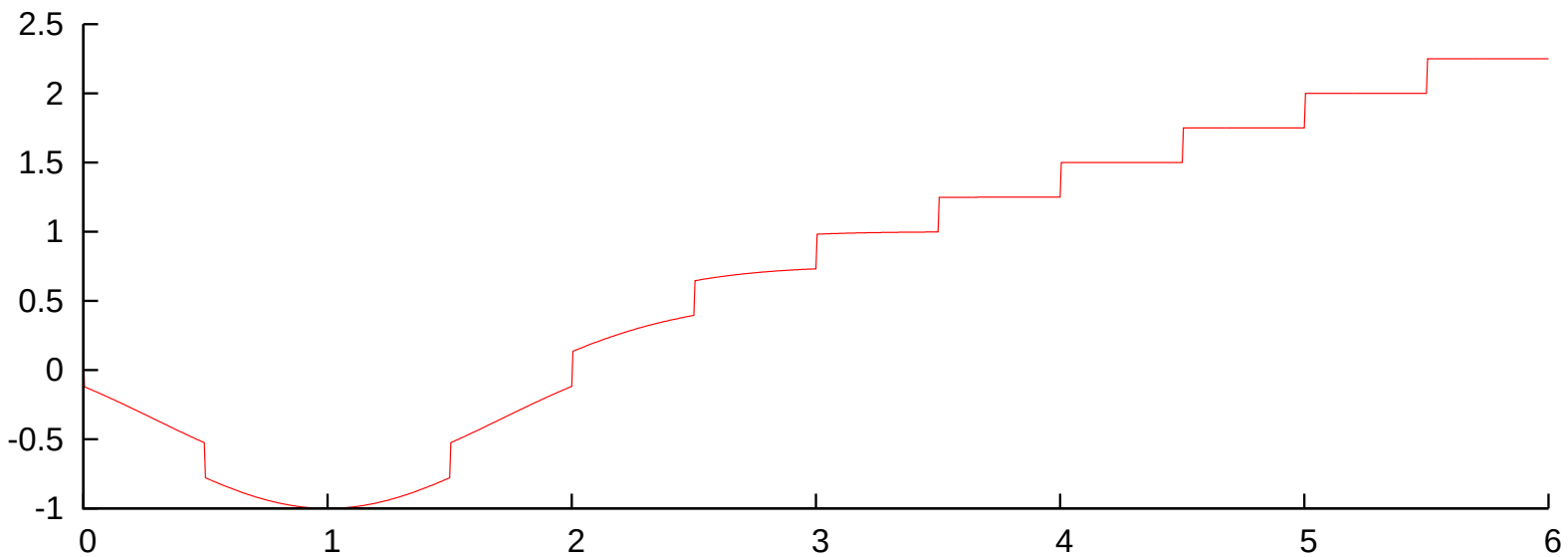
- Non-uniqueness of optimum/indeterminacy in model specification: Optimise σ, both $\sigma = -1, \sigma = 1$ give same likelihood...

Pitfalls V



- Bad Taylor-approximation: Sometimes transformation of parameters helps a bit, or it is just bad luck

Pitfalls VI



- Other numerical trouble: AVERAGE LOG likelihood, check range of optimisor, use grid search for some parameters, estimation by parts, move to simulation-based optimisation, improve starting values...

A package: Maximize

Doing Econometrics ≡ estimating models, e.g.:

1. Optimise likelihood
2. Minimise sum of squared residuals
3. Mimimise difference in moments
4. Do Bayesian simulation, MCMC

Options 1-3 evolve around

$$\hat{\theta} = \underset{\theta}{\text{argmax}} \pm f(y;\theta)$$

Inputs:

- f, use *average log* likelihood, or *average* (negative) sum-of-squares.
- Starting value θ_0
- Possibly f', analytical first derivatives of f.

Maximize I: Regression

$$y_i = X_i\beta + \epsilon_i \qquad\qquad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

ML maximises likelihood (other options: Minimise sum-of-squares, optimise utility etc):

$$\begin{aligned} L(y;\theta) &= \prod_i \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - X_i\beta)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right) \end{aligned}$$

In this case, $\theta = (\beta, \sigma^2)$

Maximize II

The `maximize` package provides features to maximize a function. So maximize

$$\texttt{AvgLnLRegr}(\theta) = LL(\theta)/N$$

First write (and test!) this function:

Listing 1: eststack_ml.ox

```
#include <oxstd.h>
#import <maximize>
// Two globals for likelihood function
static decl s_vY, s_mX;

AvgLnLiklRegr(const vP, const adLnPdf, const avScore, const amHess)
{
  adLnPdf[0]= ...;

  return !ismissing(adLnPdf[0]); // Check if not missing}
}
```

Maximize III

Function to optimize

```
/*
**  AvgLnLiklRegr(const vTheta, const adLnPdf, const avScore, const amHess)
**
**  Purpose:
**    Compute average loglikelihood of the regression model
**
**  Inputs:
**    vTheta          k+1 x 1 vector of parameters
**    s_vY            global, n x 1 vector of regressor
**    s_mX            global, n x k matrix of explanatory variables
**
**  Outputs:
**    adLnPdf         address, on output the average loglikelihood
**    avScore         (not used for now)
**    amHess          (not used)
**
**  Return value:
**    ir              boolean, TRUE if computation succeeded, FALSE otherwise
*/
```

Syntax

Function to maximize should have format

```
fnFunc(const vP, const adLnPdf, const avScore, const amHess)
```

- Choose your own logical function name
- `vP` is a $p \times 1$ COLUMN vector with parameters
- `adLnPdf` is a pointer to a variable, which on return should contain the function value, or a missing if function could not be evaluated
- `avScore` can be either 0 or a pointer. In the latter case the function should fill it in with the $p \times 1$ score vector
- `amHess` can be either 0 or a pointer, but isn't used in `MaxBFGS`
- The function should return either `TRUE` (if the calculation succeeded) or `FALSE`, if not.

Writing the likelihood

Let's do it...

To remember:

$$L(y;\theta) = (2\pi\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right)$$

$$\log L(y;\theta) = -\frac{1}{2}\left(N \log 2\pi + N \log \sigma^2 + \frac{e'e}{\sigma^2}\right)$$

In this case, $\theta = (\beta, \sigma)$ or $\theta = (\beta, \sigma^2)$.

- Start off your `adLnPdf[0]` at a missing (just so you won't forget giving it a value
- Extract your parameters from the vector, use sensible names
- Check if your parameters are valid; if not ($\sigma^2 < 0$?) return `FALSE`
- At the end, test for a `!ismissing(adLnPdf[0])`
- Careful: This routine is used often; don't do unnecessary work
- And test...

Syntax II

Call MaxBFGS according to

```
#import <maximize>
ir= MaxBFGS(fnFunc, avP, adLnPdf, amInvHess, bNumDer);
```

- `fnFunc` is the name of the function
- `avP` is a pointer to the initial vector of parameters, on return it will contain the optimal parameters
- `adLnPdf` is a pointer which on return will contain the optimal value
- `amInvHess` can be 0 or pointer to $k \times k$ matrix with initial inverse Hessian estimate; on output it gives a (bad) estimate of this matrix
- `bNumDer` is a boolean, indicating if numerical derivatives have to be used

The return value `ir` indicates the type of convergence; `MaxConvergenceMsg(ir)` returns an intelligible message-string.

Optimisation

Approach for general *criterion function* $Q(\beta)$: Write

$$Q^*(\beta) = Q(\beta_{(0)}) + g'_{(0)}(\beta - \beta_{(0)}) + \frac{1}{2}(\beta - \beta_{(0)})' H_{(0)}(\beta - \beta_{(0)})$$

$$g(\beta) = \frac{\partial}{\partial \beta} Q(\beta)$$

$$H(\beta) = \frac{\partial^2}{\partial \beta \partial \beta'} Q(\beta)$$

Optimise approximate $Q^*(\beta)$:

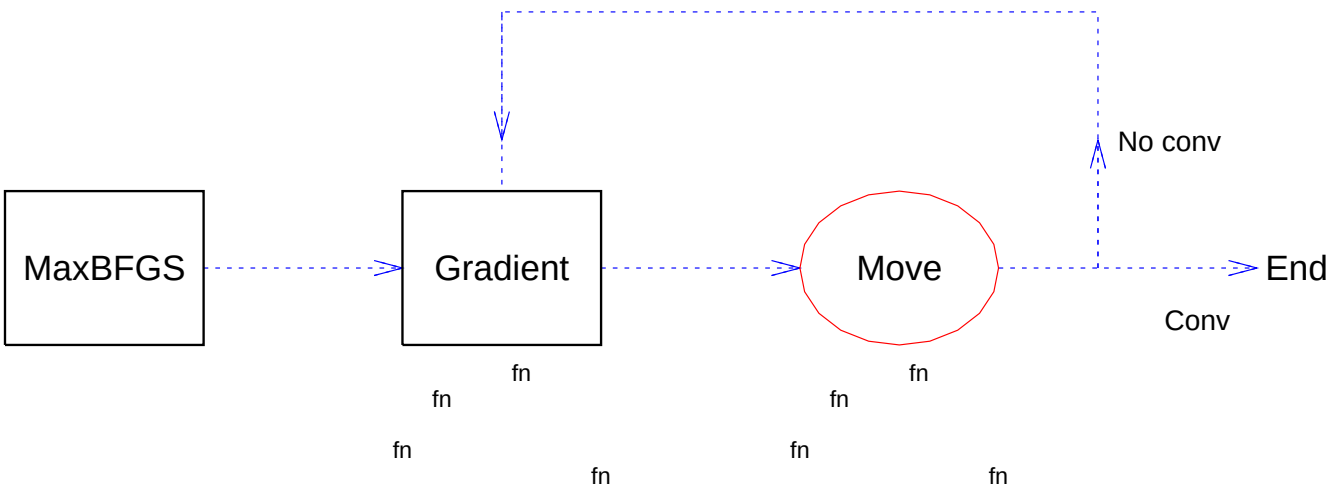
$$g_{(0)} + H_{(0)}(\beta - \beta_{(0)}) = 0 \qquad \text{First order conditions}$$

$$\Leftrightarrow \beta_{(1)} = \beta_{(0)} - H_{(0)}^{-1} g_{(0)}$$

and iterate into oblivion.

MaxBFGS: Program flow

MaxBFGS follows quasi-Newton method according to Broyden, Fletcher, Goldfarb and Shanno (BFGS).
Can use either numerical or analytical first derivatives. Analytical derivatives are more robust, exact, quicker.



Maximize: Average

Why use average loglikelihood?

1. Likelihood function $L(y;\theta)$ tends to have tiny values $\rightarrow$ possible problem with precision
2. Loglikelihood function $\log L(y;\theta)$ depends on number of observations: Large sample may lead to |large LL|, not stable
3. Average loglikelihood tends to be moderate in numbers, well-scaled...

Better from a numerical precision point-of-view.
Warning:

Take care with score and standard errors (see later)

Maximize: Precision

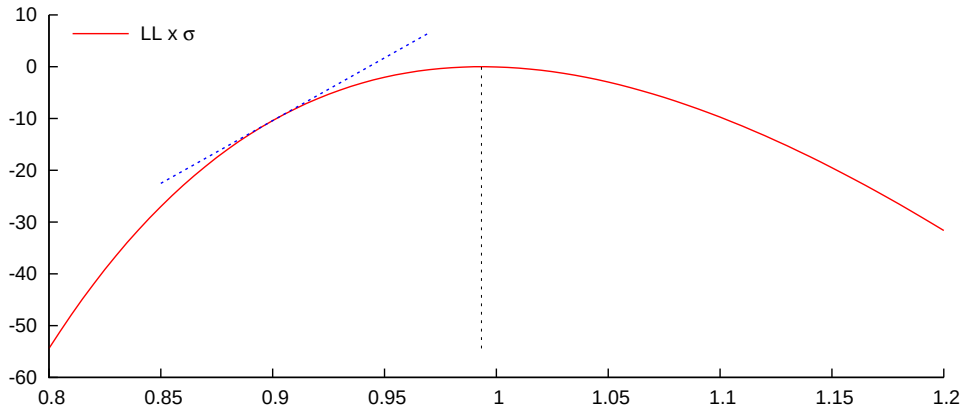
Strong convergence is said to occur if (roughly):

1. $|q_j^i \theta_j^i| \leq \epsilon_1, \forall j$, with q_j^i the jth element of the score at θ^i, at iteration i: Scores are relatively small.
2. $\frac{|\theta_j^i - \theta_j^{i-1}|}{|\theta_j^i|} \leq 10\epsilon_1$: Change in parameter is relatively small

Note: This also depends on the scale of your parameters...
Preferably $\theta \approx 1$, not $\theta \approx 1e-15$!

Adapt the precision with `MaxControlEps(const dEps1, const dEps2);`,
default is `dEps1= 1e-4, dEps2= 5e-3`.

Maximize: Scores



Optimising ≡ 'going up'
≡ finding gradient.

Numerical gradient, for small h:

$$f'(\theta) = \frac{\partial f(\theta)}{\partial \theta} \approx \frac{f(\theta + h) - f(\theta)}{h} \approx \frac{f(\theta + h) - f(\theta - h)}{2h}$$

Function evaluations: $2 \times \dim(\theta)$

Preferred: Analytical score $f'(\theta)$

Maximize: Scores II

```
AvgLnLiklRegr(const vTheta, const adLnPdf, const avScore, const amHess)
{
  ...
  if (isarray(avScore))      // Check if score is requested
    {
      avScore[0]= ???;       // Compute the score, in whatever way
    }
}
```

- Only compute the score when requested
- Test if `avScore` is an *array* (as this looks similar to an address)
- Or test if `avScore` is non-zero: `if (avScore) ...` , should do the same thing
- Work out vector of scores, of same size as θ.
- DEBUG! Check your score against `Num1Derivative()`

Maximize: Scores IIb

- ...
- DEBUG! Check your score against `Num1Derivative()`

```
#import <maximize>
...
AvgLnLiklRegr(vTheta, &dLnPdf, &vS1, 0);      // Compute analytical score
Num1Derivative(AvgLnLiklRegr, vTheta, &vS2); // Compute numerical score

print ("Theta and scores: ",
       vTheta~vS1~vS2~(vS1-vS2));             // Compare scores
```

Don't ever forget debugging this
(goes wrong 100% of the times...)

Maximize: Scores III

Let's do it...

To remember:

$$f(y;\theta) = -\frac{1}{2}\left(\log 2\pi + 2\log\sigma + \frac{e'e}{N\sigma^2}\right)$$

$$e = y - X\beta$$

$$\frac{\partial f(y;\theta)}{\partial \beta} = \ldots$$

$$\frac{\partial f(y;\theta)}{\partial \sigma} = \ldots$$

- In this case, it matters whether $\theta = (\beta, \sigma)$ or $\theta = (\beta, \sigma^2)$!
- Find score of AVERAGE loglikelihood

Standard deviations

Given a model with

$$\begin{aligned} \mathcal{L}(Y;\theta) &\qquad \text{Likelihood function} \\ l(Y;\theta) = \log \mathcal{L}(Y;\theta) &\qquad \text{Log likelihood function} \\ \hat{\theta} = \text{argmax}_\theta l(Y;\theta) &\qquad \text{ML estimator} \end{aligned}$$

what is the vector of standard deviations, $\sigma(\hat{\theta})$?
Assuming correct model specification,

$$\Sigma(\hat{\theta}) = -H(\hat{\theta})^{-1}$$

$$H(\hat{\theta}) = \left. \frac{\delta^2 l(Y;\theta)}{\delta\theta\delta\theta'} \right|_{\theta=\hat{\theta}}$$

SD2: Average likelihood

For numerical stability, optimise *average* loglikelihood.

For regression model, e.g. the stackloss model,

$$l(Y;\theta) = -\frac{(y - X\beta)'(y - X\beta)}{2\sigma^2} - N \log 2\pi\sigma^2 + c$$

$$\bar{l}(Y;\theta) = -\frac{(y - X\beta)'(y - X\beta)}{2N\sigma^2} - \log 2\pi\sigma^2 + c'$$

$$H_{\bar{l}} \equiv \frac{\delta^2 \bar{l}(Y;\theta)}{\delta\theta\delta\theta'} = \frac{1}{N}\frac{\delta^2 l(Y;\theta)}{\delta\theta\delta\theta'} \qquad \hat{\Sigma}(\hat{\theta}) = \frac{1}{N}(-H_{\bar{l}})^{-1}$$

```
mS2= 0;
if (Num2Derivative(AvgLnLiklRegr, avP[0], &mH))
  mS2= invertgen(-mH, 30)/iN,

avS[0]= sqrt(diagonal(mS2)');

print ("MaxBFGS returns ", MaxConvergenceMsg(ir),
       " with LL= ", adLL[0]*iN,
       "at parameters ",
       "%c", {"Par", "Std"}, avP[0]~avS[0]);
```