Fair agreements for sharing international rivers with multiple springs and externalities[1]

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Abstract

We consider the problem of sharing water among agents located along a river. Each agent's benefit depends on the amount of water consumed. An allocation of water is efficient when it maximizes total benefits. To sustain an efficient water allocation the agents can compensate each other by monetary transfers. Every water allocation and transfer schedule yields a welfare distribution, where an agent's utility equals its benefit plus (possibly negative) monetary transfer. The problem of finding a fair welfare distribution can be modeled by a cooperative game. We consider river situations with satiable agents and multiple springs. We propose the class of weighted hierarchical solutions, including the downstream incremental solution of Ambec and Sprumont (2002), as a class of solutions satisfying the 'Territorial Integration of all Basin States' principle for sharing water of international rivers. When all agents have increasing benefit functions, every weighted hierarchical solution is core-stable. In case of satiation points, every weighted hierarchical solution is independent of the externalities.

Keywords: Water allocation, river game, externality, core, hierarchical outcome.

JEL code: C71, D62, H23

1 Introduction

The aim of this paper is to introduce and analyze rules for a fair distribution of welfare resulting from allocating the water flow of a river among the agents (countries, cities, firms) along the river. We consider the case of an international river, where the agents are countries located from upstream to downstream along the river, and water flows possible from the inland into the river on every agent's territory.[1] In case of common aquifers or a boundary river, i.e., a river that forms a common border, all agents have equal access to the water. In case of an international river the one-directionality of the water flow has to be taken into account when allocating the water to the agents: water inflow at the territory of downstream agents can not be allocated to upstream agents, because the water flows from upstream to downstream. Typically, a welfare maximizing allocation of the water flow among the agents requires that some agents do not consume the total inflow on their own territory. When downstream agents have higher marginal benefits than upstream agents, welfare maximization requires that upstream agents let pass some of their inflow to downstream agents. However, when the benefits from the consumption of water are increasing, giving up water harms the upstream agents. Therefore monetary compensations from the downstream agents to the upstream agents maybe helpful to make a welfare maximizing allocation implementable. International law states that the nations involved should mutually agree on sharing the river through negotiations, but it is left in the middle to what extent unilateral decisions can be made in the absence of agreement. Moreover, such negotiations are often deadlocked, because almost all governments in water stressed regions became aware of water issues after having experienced serious shortages. Unless politics either deepens or broadens the water agenda, as in e.g. Bennett, Ragland and Yolles (1998), the situation is most likely to stay put or might even deteriorate ending in conflict.

For a river with one spring and one sink, i.e., the agents are successively located from the most upstream agent at the spring of the river to the most downstream agent at the sink, Ambec and Sprumont (2002) model the welfare distribution problem as a cooperative game on the set of agents (countries). A solution assigns to every allocation problem a distribution of the attainable welfare. Ambec and Sprumont (2002) propose and axiomatize the so-called *downstream incremental solution.* Suppose there are n agents along the river and let the agents be numbered by 1 to n successively from upstream to downstream. Then the welfare assigned to agent i by the downstream incremental solution is equal to the gain in maximal welfare if i joins its upstream agents 1 to $i-1$. This is equal to the total welfare that can be attained by the coalition of all (upstream) agents from 1 to i (by

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[1] The results also apply when the agents are cities or firms along the river within one country.

allocating optimally the water inflow on their own territory among themselves, under the restriction of the one-directionality of the river water) minus the total welfare that can be attained by the coalition consisting of the agents from 1 to $i-1$.

In Ambec and Sprumont (2002) it is assumed that the benefit that an agent derives from water consumption is strictly increasing in the amount of water. Under this assumption only coalitions of consecutive agents will form. When, for instance, a coalition consists of an upstream consecutive part and a downstream consecutive part with in-between some agents outside the coalition, then every water flow sent from the upstream part to the downstream part would immediately be taken by the in-between agents. In Ambec and Ehlers (2008) the assumption of increasing benefit functions is weakened by allowing for satiable agents. The existence of satiation points has serious consequences, because now also coalitions of non-consecutive agents might form. When every outside agent that is in-between two consecutive parts has a satiation point, then it might be profitable for the upstream part of a coalition to pass water to the downstream part. Although some of this flow might be taken by the in-between agents, these agents will only take water up to their satiation points. When the flow is big enough, cooperation between the upstream part and the downstream part could still be profitable. This behavior might cause positive *externalities* on the agents between the two parts since the welfare of a coalition of consecutive agents along the river might be affected positively from cooperation between agents upstream and downstream of the coalition.[2] However, since the downstream incremental distribution only depends on the highest attainable welfare levels of the coalitions of agents from 1 to i, for $i = 1, \ldots, n$, this solution is *externality-free*, i.e., it only depends on the welfare levels of coalitions that do not experience externalities from agents outside the coalition.

In this paper we derive, within the framework of a cooperative game, fair welfare distributions for international rivers with multiple springs (i.e. there are side rivers merging into one mainstream) by taking into account the so-called *TIBS-principle* that can be used in disputes on water allocation within an international river basin. As in Ambec and Ehlers (2008), we allow for satiable agents. The novelty of this paper is threefold. First, we allow for multiple springs. Second, using the TIBS-principle we introduce and characterize a new class of solutions. Third, we show that in case of satiable agents this class is externality-free.

Concerning the first novelty, models that have been developed up to this point deal only with single-stream rivers.[3] However, international river basins with multiple springs in which there is (potential) conflict over the distribution of water include some of the

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[2] This type of externality is consistent with the definition given by e.g. Meade (1973): ‘An external economy (diseconomy) is an event which confers an appreciable benefit (inflicts an appreciable damage) on some person or persons who were not fully consenting parties in reaching the decision or decisions which led directly or indirectly to the event in question’.

[3] Khmelnitskaya (2010) is a notable exception, but does not take account of externalities.

worlds largest basins. Since multiple springs might lead to a substantial improvement for downstream agents in negotiations on the water, it is useful to study such river systems. We discuss this in more detail in Subsection 2.1.

Concerning the second novelty, in Kilgour and Dinar (1995) several principles on water rights have been listed that can be taken into account to prevent or resolve water disputes within an international river basin. According to these authors, the principle of Territorial Integration of all Basin States (TIBS) accords to each agent 'equal' use, without regard to its contribution to the flow. The TIBS principle does not consider any country the owner of the water, but instead states that the water belongs to all countries together, no matter where it enters the river. Taking the one-directionality into account, a possible interpretation is that the inflow at some country should be shared equally by this country and all its downstream countries. The TIBS principle is also known as the principle of community of interests in the waters or the principle of common management and can, alternatively, be described as follows: 'the water belongs to all basin states combined, no matter where it enters the river, and each state is entitled to a reasonable and equitable share in the optimal use of the water', see e.g. Lipper (1967) or McCaffrey (2001). According to this description, the principle requires that (i) the water is assigned in such a way that the total welfare of all countries is maximized (optimal use) and (ii) each country gets a (reasonable and equitable) share in the total welfare resulting from an optimal assignment. In this paper we apply the TIBS principle in the following way to a river basin with multiple springs and satiable agents. Suppose that, for one reason or another, the agents along a river with multiple springs are cooperating in two separate coalitions as follows. For some agent, say i, one of the coalitions consists of agent i and all its upstream agents, and the other coalition is its complement, i.e., consists of all other agents. For instance this happens when agent i is not willing to cooperate with its unique downstream neighbor. The question that then can be asked is: how should the gain in total welfare that is created when the two coalitions join together into one coalition of all agents, be divided among the agents? Evidently, we can ask this question for every single agent i (except the unique most downstream agent). The TIBS principle provides us with an answer. Let there be for each agent a nonnegative number, its weight, with sum over all agents equal to one. Then we interpret the TIBS principle by requiring that, for each agent i, the gain in welfare, that is created by merging i's upstream coalition and its complement, is divided among the two coalitions proportional to the sum of the weights of the agents in these two coalitions. We will show that, for every specific vector of weights, this requirement characterizes a particular distribution of the total welfare resulting from a welfare maximizing water allocation (optimal use). The extreme case where the most downstream agent has weight equal to one yields the downstream incremental welfare distribution, the special case that all weights are taken to be equal yields a welfare distribution that can be seen as taking the average of so-called *hierarchical outcomes* as introduced by Demange (2004).

Concerning the third novelty, we generalize the TIBS principle to allow for satiable agents, and show that every solution satisfying this generalized TIBS principle is *externality-free*, meaning that the welfare distribution only depends on the worths of coalitions that do not experience externalities from the cooperation structure on the set of agents outside the coalition.

The paper is organized as follows. Section 2 puts the international river basin problem within the context of game theory and provides the cooperative game model for international rivers with multiple springs. In Section 3 we discuss several principles for allocating water in international rivers with multiple springs, but without satiated agents, and specify these principles into axioms for solutions in our game model. We characterize the solutions that are determined by these principles. In Section 4 we discuss the generalized case with satiable agents. Finally, Section 5 contains concluding remarks.

2 River systems with multiple springs

2.1 Research problem and approach

Coalition formation, the division of gains within coalitions, unilateral decisions prior to the negotiations and incentive compatibility to sustain an agreement, belong traditionally to the realm of game theory. This is also recognized by global institutions involved in river management such as the World Bank, see for instance Carraro, Machiori and Sgobbi (2005a,b). Many researchers in economics and game theory have addressed water issues, see for instance Dinar, Ratner and Yaron (1992) and Dinar *et al.* (2005) for extensive surveys. A lot of this research focuses on the problem of allocating water in which a common resource has to be shared by several users, for instance allocating water in case of boundary rivers or in case of common groundwater aquifers. In contrast to such common pool situations, a feature peculiar to international rivers is the one-directionality of the water flow, imposing in some sense dominance of upstream agents over downstream agents. International water law, as laid down in the Helsinki Rules of 1966 and the UN Convention on the Law of the Non-Navigational Uses of International Watercourses of 1997, does neither recognize claims by upstream countries on the water caught on its territory, nor downstream nations claims of historical rights, which makes the application of negotiation theory even more difficult.

We consider international river basins with multiple springs. Examples of such basins in which there is (potential) conflict over the distribution of water include some of the worlds largest basins: the Amazon basin, the Euphrates and Tigris basin, the Ganges-

Brahmaputra-Meghna basin, the Mekong basin and the Nile basin. In the Amazon basin water shortage is currently no real issue, but in 2005 parts of the basin experienced severe droughts. Similar droughts in the future could lead to friction between states sharing the basin. In the Euphrates and Tigris basin, Turkey has more than once been accused by Syria and Iraq of depriving them of water. In the Ganges-Brahmaputra-Meghna basin, the most heavily populated river basin in the world, water shortages in some places are getting worse with sections of the river running dry for parts of the year. The building of dams by upstream states in the Mekong basin (China and Thailand) is causing low flows in downstream states (Cambodia and Vietnam), that are completely dependent on the river for food and the majority of their developing economies. The best example of disagreement in an international river basin with multiple springs is probably that of the Nile river. The Nile river, generally regarded as the longest river in the world, has two main tributaries, the White Nile and the Blue Nile, and runs through the eleven territories of Tanzania, Burundi, Rwanda, Democratic Republic of the Congo, Uganda, South Sudan, Sudan, Ethiopia, Eritrea, and Egypt. Historically, Egypt (the most downstream country) has claimed most of the rights over the use of water from the Nile. In recent decades, however, water sharing disputes with upstream countries, including Uganda, Sudan and Ethiopia, have erupted about the Egyptian domination of Nile resources. In 1999 the Nile Basin Initiative was launched with the goals of developing the river in a cooperative manner, sharing substantial socioeconomic benefits, and promoting regional peace and security.

The examples above show that analyzing rivers with multiple springs has many important applications. Since multiple springs have a great impact on the position of the agents in water negotiations, it is therefore important to study such river structures. To see the limited use of single stream river games, consider the following example. In a single stream river basin with one upstream agent and one downstream agent, the two agents are completely dependent on each other when it comes to the trade of river water. The upstream agent holds water the downstream agent might want and the downstream agent holds money the upstream agent might want. In a river basin with two springs, at the territories of two different countries, that merge together to one stream at the territory of a single downstream agent, the position of the latter agent in negotiations is completely different. While the downstream agent still holds money the upstream agents might want, there are now two upstream agents that hold water the downstream agent might want. Hence, there are now (possibly) two suppliers of the good 'water' and (possibly) only one agent that demands it. Therefore, in negotiations this downstream agent is in a much better position than a downstream agent having only one upstream neighbor and thus ignoring multiple springs could be a serious shortcoming of the analysis.

The international river problem is studied in for instance Kilgour and Dinar (1995, 2001), Bennett, Ragland and Yolles (1998) and Supalla *et al.* (2002). We follow the approach of Ambec and Sprumont (2002) and Ambec and Ehlers (2008) who consider the problem to find a 'fair' distribution of the welfare resulting from allocating the water flows of an international river with one spring (and one sink) to the agents (e.g. firms, cities, countries) located along the river from upstream to downstream. They modeled this problem as a cooperative game on the set of agents (countries) along the river. In this paper we extend their model to rivers with multiple springs.

A *cooperative game with transferable utility in characteristic function form*, or simply a TU-game, is a pair (N, v), where $N = \{1, ..., n\}$ is a finite set of n players, and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. Since we take the player set N to be fixed, we represent a TU-game by its characteristic function v and we denote the collection of all TU-games on N by $\mathcal{G}^N$.

2.2 Cooperative game model of international river basins

To describe a river system with several side-rivers which originate at different springs and that merge together to one river, let $N = \{1, \ldots, n\}$ be the set of players representing the agents, in the sequel also called countries, along the river. Further, each spring is identified by an agent, i.e., we consider the most upstream agent along a side river as its spring. Every agent has precisely one downstream neighbor (except the final most downstream agent), but agents can have multiple upstream neighbors, namely in case two (or more) streams merge at the territory of an agent.[4] We denote the number of springs by s and denote $O = \{1, \ldots, s\}$ as the set of agents located at some spring, i.e., agent j, $j \in \{1, \ldots, s\}$, is located at spring j. Let $n > s$ be the total number of agents. We index the (unique) most downstream agent by n. For agent k, let U^k denote the set of upstream neighbors of k. Every agent $i \in N\backslash\{n\}$ is in exactly one U^k for some $k \in N$. Notice that the structure of the river system is fully determined by the n-tuple of sets $(U^k)_{k \in N}$, with $U^k = \emptyset$ if and only if $k \in O$. We denote this n-tuple by $\mathcal{U} = \{U^k | k \in N\}$.[5] A pair $(N, \mathcal{U})$ consisting of a set of agents N and a river structure $\mathcal{U}$ is then called a *river system*. Notice that for a river with a single spring and agents numbered successively from upstream to downstream by 1 to n, the river system $(N, \mathcal{U})$ is given by $U^1 = \emptyset$ and $U^k = \{k - 1\}$ for $k = 2, \ldots, n$.

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[4] In this paper we assume that at the territory of an agent there is at most one spring. When there are two springs at the territory of an agent, we consider this as one single spring and so we do not consider systems having two (or more) springs at the territory of one agent that go to different downstream agents.

[5] Formally, for a set N of agents, a river structure is a collection $\mathcal{U} = \{U^k | k \in N\}$, such that (i) for some $1 \leq s < n$, $U^k \neq \emptyset$ if and only if $k > s$, (ii) $U^k \cap U^h = \emptyset$ for all $k, h \in N,\ k \neq h$, and (iii) $\cup_{k \in N} U^k \subset N$.

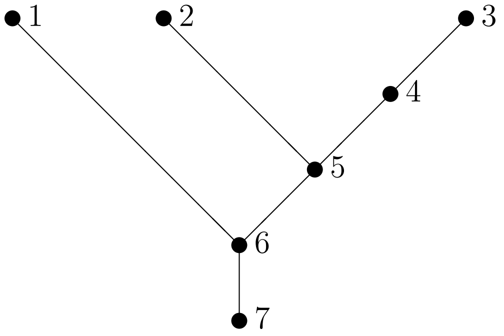

Figure 1: River structure from Example 2.1

Example 2.1 Let $(N,\mathcal{U})$ represent a river system with $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $U^1 = U^2 = U^3 = \emptyset$, $U^4 = \{3\}$, $U^5 = \{2, 4\}$, $U^6 = \{1, 5\}$ and $U^7 = \{6\}$, see Figure 1. So, $O = \{1, 2, 3\}$ is the set of springs and $n = 7$ is the (unique) most downstream agent. The two rivers originating at 2 and 3 merge together at agent 5 and then this stream merges together at agent 6 with the side river originating at agent 1. □

For $k \in N$, let P^k denote the set of all agents upstream of k, including k itself. Clearly, (i) $U^k \subseteq P^k \setminus \{k\}$ for every $k \in N$, (ii) $P^k = \{k\}$ and $U^k = \emptyset$ if $k \in O$, and (iii) $P^n = N$. Further, denote $N_k = N \setminus P^k$, i.e., N_k is the complement of the set P^k consisting of the set of agents not in P^k. Thus, N_k contains all agents downstream to agent k and also all springs $j \in O \setminus P^k$ that are not upstream of k and all agents downstream of these springs. Notice that for every agent k, both the $|P_k|$-tuple[6] $(U^i \cap P^k)_{i \in P^k} = (U^i)_{i \in P^k}$ and the $|N_k|$-tuple $(U^i \cap N_k)_{i \in N_k}$ are also river structures. So, the pairs $(P_k, (U^i)_{i \in P^k})$ and $(N_k, (U^i \cap N_k)_{i \in N_k})$ are sub-river systems on the sets P^k, respectively N_k. Finally, let Q_k denote the set of all agents downstream to agent k, including k itself, and for $k \neq n$, let d_k be the unique downstream neighbor of k. Taking $k = 5$ in Example 2.1, $P^5 = \{2, 3, 4, 5\}$, $N_5 = \{1, 6, 7\}$, $Q_5 = \{5, 6, 7\}$ and $d_5 = 6$. Further, the sub-river system $(N_5, (U^i \cap N_5)_{i \in N_5})$ is given by the river structure $U^1 \cap N_5 = \emptyset$, $U^6 \cap N_5 = \{1\}$ and $U^7 \cap N_5 = \{6\}$.

To complete the description of the river problem, let $e_i \geq 0$ be the inflow of water on the territory of agent i, $i = 1, \ldots, n$. Finally, following Ambec and Sprumont (2002), every agent is assumed to have a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$, where $t_i \in \mathbb{R}$ is a monetary compensation to agent i, $x_i \in \mathbb{R}_+$ is the amount of water allocated to agent i, and $b^i \colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous function yielding the benefit $b^i(x_i)$ to agent i of the consumption x_i of water. Ambec and Sprumont (2002) make the following assumption.

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[6] For a set A, $|A|$ denotes the number of elements in A.

Assumption 2.2 *In the river game, every benefit function* $b^i\colon \mathbb{R}_+ \to \mathbb{R}$ *is a strictly increasing and strictly concave function, which is differentiable for* $x_i > 0$ *with derivative going to infinity as* x_i *tends to zero.*

Because of the one-directionality of the water from upstream to downstream, the water inflow downstream of some agent can not be allocated to this agent. Thus, every agent can be assigned at most the water inflow at the territories of himself and his upstream agents. Hence, a vector $x \in \mathbb{R}^n_+$ assigning an amount of water x_i to agent i, $i = 1, \ldots, n$, is a *water allocation* only if it satisfies the feasibility restrictions

$$\sum_{i \in P^j} x_i \leq \sum_{i \in P^j} e_i, \quad j = 1, \ldots, n,$$

i.e., for every agent j, the sum of the water assignments to agent j and all its upstream agents is at most equal to the sum of the inflows at j and all its upstream agents. A water allocation x yields *total welfare* $\sum_{i=1}^n b^i(x_i)$. A *compensation scheme* $t \in \mathbb{R}^n$ gives a monetary compensation t_i to agent i, $i = 1, \ldots, n$, under the restriction

$$\sum_{i=1}^n t_i \leq 0.$$

So, the sum of all positive compensations (agents that receive money) is at most equal to the absolute value of the sum of all negative compensations (agents that have to pay). A *welfare distribution* is a pair (x, t) of a water allocation x and a compensation scheme t yielding utility $u^i(x_i, t_i) = b^i(x_i) + t_i$ to every agent i, $i = 1, \ldots, n$. A welfare distribution is *Pareto efficient* if no water and no money is wasted. So (x, t) is Pareto efficient if and only if $x \in \mathbb{R}^n_+$ maximizes the welfare maximization problem

$$\max_{x_1,\ldots,x_n} \sum_{i=1}^n b^i(x_i) \text{ s.t. } \sum_{i \in P^j} x_i \leq \sum_{i \in P^j} e_i, \quad j = 1, \ldots, n, \text{ and } x_i \geq 0, \; i = 1, \ldots, n, \qquad (2.1)$$

and the compensation scheme is *budget balanced*:

$$\sum_{i=1}^n t_i = 0.$$

By Assumption 2.2, the maximization problem (2.1) has at least one solution and every solution x^* yields the same attainable maximum welfare $\sum_{i=1}^n b^i(x^*_i)$. In the sequel, let $W(N, \mathcal{U}, e, b)$ denote the maximum welfare for river system $(N, \mathcal{U})$ with vector $e \in \mathbb{R}^n_+$ of inflows and benefit functions $b = (b^i)_{i \in N}$. For a solution x^*, a Pareto efficient welfare distribution (x^*, t) yields payoffs (utilities)

$$z_i = b^i(x^*) + t_i, \quad i = 1, \ldots, n,$$

with sum of payoffs equal to the total welfare $W(N,\mathcal{U},e,b)$.

The problem to find a 'fair' distribution of the Pareto efficient total welfare can be modeled by the following TU-game (N,v). Obviously, the worth $v(N)$ is given by $v(N) = W(N,\mathcal{U},e,b)$. Further, given a single spring river with agents numbered successively by 1 to n from upstream to downstream, for any pair of agents i,j with $j > i$, it holds that the water inflow entering the river before the upstream agent i can only be allocated to the downstream agent j if every agent h between agents i and j cooperates. Otherwise, since every benefit function b^h is strictly increasing in x_h, every agent h between i and j can increase its utility by taking the flow from i to j for its own use. Hence, for a single spring river, a coalition T is admissible if and only if T is *consecutive*, i.e., $T = \{i, i+1, \ldots, j\}$ for some $i,j \in N,\ i \leq j$. For convenience we denote for any pair of agents i,j with $j > i$, the coalition $\{i, i+1, \ldots, j\}$ of consecutive agents by $[i,j]$.

In case of a river system $(N,\mathcal{U})$ with multiple springs, a coalition of agents S can cooperate together when (i) there exists a $k \in S$ such that $S \subseteq P^k$, and (ii) for every $i \in S \setminus \{k\}$, every agent between i and k is also in S. Condition (i) means that agents on two branches can not cooperate if they do not have a common most downstream agent, for instance in Example 2.1 the two upstream branches $\{1\}$ and $\{3,4\}$ can not benefit from cooperation in the coalition $\{1,3,4\}$. Condition (ii) generalizes the notion of a coalition of consecutive agents to the case of multiple springs: it implies that when j cooperates with an upstream agent i, every agent on the branch between i and j also cooperates. In Example 2.1, agents 2 and 6 can only cooperate when also 5 agrees. We say that a coalition S is *connected* when S satisfies (i) and (ii). Modeling this situation as a TU-game, the worth $v(S)$ of a connected coalition S is given by

$$v(S) = \sum_{h\in S} b^h(x_h^S) \quad \text{where} \quad x^S = (x_h^S)_{h\in S} \quad \text{solves}$$

$$\max_{\{x_h \geq 0 | h\in S\}} \sum_{h\in S} b^h(x_h) \quad \text{s.t.} \quad \sum_{i\in P^j\cap S} x_i \leq \sum_{i\in P^j\cap S} e_i, \quad j\in S, \ \text{and} \ x_i \geq 0, \ i\in S. \tag{2.2}$$

For any other (non-connected) coalition S, the worth $v(S)$ is equal to the sum of the worths of its maximally connected subsets.[7]

By definition the set P^k is connected for every k, but at an agent k with at least two upstream neighbors (i.e. k is an agent at which at least two rivers merge together) the set $P^k \setminus \{k\}$ is not connected (in Example 2.1 the set $P^5 \setminus \{5\} = \{2,3,4\}$ is not connected, but it contains two maximal connected subsets: $\{2\}$ and $\{3,4\}$). For every $k \in N$ it holds

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[7] A subset T of S is maximal connected in S if T is connected in the sub-river system $(S, (U^k \cap S)_{k\in S})$ and $T \cup h$ is not connected in this sub-river system for any $h \in S \setminus T$.

that

$$v(P^k \setminus \{k\}) = \sum_{j \in U^k} v(P^j).$$

The triple $(N, \mathcal{U}, v)$ describes a river situation with multiple springs. When we take the river system $(N, \mathcal{U})$ as given, we denote the collection of all characteristic functions v obtained from river situations on $(N, \mathcal{U})$ with benefit functions satisfying Assumption 2.2 by $\mathcal{G}^{(N, \mathcal{U})}$.

To obtain a solution for river problems, one can apply any game solution to the associated cooperative game defined above. A payoff vector of a game $v \in \mathcal{G}^N$ is a vector $y \in \mathbb{R}^n$ such that y_i is the payoff assigned to player $i \in N$ in game v. The *core* (Gillies, 1953) of a game $v \in \mathcal{G}^N$ is the set of all efficient payoff vectors that are stable in the sense that no coalition $S \subset N$ can do better by deviating from the grand coalition N and realizing its own worth $v(S)$, i.e.

$$\text{Core}(v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}.$$

The core might be empty.[8]

For a collection of games $\mathcal{G} \subseteq \mathcal{G}^N$, a solution is a function f on $\mathcal{G} \subseteq \mathcal{G}^N$ that assigns precisely one payoff vector $f(v) \in \mathbb{R}^n$ to every $v \in \mathcal{G}$. We can now apply any solution for cooperative games to find a welfare distribution in international river basin problems. Notice that a solution f assigns payoff vector $z = f(v) \in \mathbb{R}^n$ to game $v \in \mathcal{G}^{(N, \mathcal{U})}$. When $\sum_{i=1}^n z_i = v(N)$, then z can be implemented by a Pareto efficient welfare distribution (x^*, t), with $t_i = z_i - b^i(x_i^*)$, $i = 1, \ldots, n$. The fairness of such a distribution depends on the properties of the solution. A minimal requirement is that the solution should be core-stable, thus $f(v) \in \text{Core}(v)$ for every $v \in \mathcal{G}^{(N, \mathcal{U})}$. Notice that core-stability implies that the sum of the payoffs is equal to $v(N)$ and thus Pareto efficiency.

3 Welfare distribution in international river basins

3.1 Absolute Territorial Sovereignty and Unlimited Territorial Integrity

In Kilgour and Dinar (1995) several principles have been proposed to prevent or resolve disputes on water allocation within an international river basin. Since there is no binding

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[8] It is well-known that the core of a game is nonempty if and only if v is balanced, see e.g. Bondareva (1963) or Shapley (1967).

international law governing the allocation of water in international rivers, these principles, in combination with two legal texts called the 1966 Helsinki Document (see Kilgour and Dinar) and the 1997 UN convention, are the only guidelines that are available in determining a 'fair' welfare distribution and thus a 'fair' solution for the class of cooperative TU-games on river systems with multiple springs.[9]

Two principles from international river law used by Ambec and Sprumont (2002) are the principle of Absolute Territorial Sovereignty (ATS) and the principle of Unlimited Territorial Integrity (UTI).

The ATS principle (also known as the Harmon doctrine), states that an agent has absolute sovereignty over the area of any river basin on its territory. This principle favors upstream agents in that it allows agents to use any water that flows into the river on their territory without taking into account what consequences this might have for the countries downstream to it. For a solution f on the class $\mathcal{G}^{(N,\mathcal{U})}$ of river games, this principle implies that for every connected coalition of agents, the agents can secure themselves at least the welfare level that can be reached by allocating optimally the water that they control amongst themselves (given the one-directionality of the flow). So, for every connected coalition S it should hold that $\sum_{i \in S} f_i(v) \geq v(S)$. Then this condition also holds for every not connected subset, because for such a set the worth $v(S)$ is equal to the sum of the worths of its maximally connected subsets. So, the ATS property results in the requirement that the welfare distribution should be core-stable.

In contrast, the UTI principle favors downstream agents by stating that every country has the right to demand that the natural flow of the river is not altered by its upstream countries. The UTI principle is often criticized, because it leads to the conflicting consequence that the inflow at the territory of some country can be claimed by this country and all its downstream countries. However, for the single spring river game, Ambec and Sprumont (2002) apply the UTI principle to require that the welfare distribution should satisfy the property that no coalition S should get a total welfare above its *aspiration level*, defined as the highest sum of all benefits over the agents in S that can be obtained by an optimal allocation among its own members of all the water inflows at all agents along the river from 1 to s, where s is the most downstream agent of coalition S. So, the aspiration level is the welfare level that can be obtained when the agents in S can also use the water inflows of the agents not in S, but upstream to the most downstream member of S. Applying this to the upstream coalition $[1, j] = \{1, \ldots, j\}$ of consecutive agents from 1 to j, this *aspiration level fairness* property requires that the solution of the cooperative TU-game gives a total payoff to coalition $[1, j]$ at most equal to the aspiration level upperbound

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[9] These principles can, of course, also be used when the agents in the model are not countries but, for instance, firms or cities.

$v([1,j])$. On the other hand, core-stableness requires that coalition $[1,j]$ receives at least $v([1,j])$. Therefore, for every upstream coalition $[1,j]$, $j = 1,\dots,n$, core-stableness and aspiration level fairness together imply that the total payoff to the agents in coalition $[1,j]$ should be equal to $v([1,j])$. For single spring rivers, this uniquely determines a welfare distribution, being the *downstream incremental welfare distribution*, that is given by

$$f_1^d(v) = v(1) \text{ and } f_i^d(v) = v([1,i]) - v([1,i-1]), \quad i = 2,\dots,n,$$

and assigns to every agent its contribution to the total welfare when it enters the coalition consisting of its upstream agents.[10] In the sequel, we refer to the solution f^d that assigns this welfare distribution to any single spring river game as the *downstream incremental solution*. Since under Assumption 2.2 a single spring river game is convex[11] (see Ambec and Sprumont 2002), and every marginal vector of a convex game is in the core of the game, it follows that the downstream incremental solution is core-stable.[12]

The downstream incremental welfare distribution has the property that for every $i < n$, the total payoff of the agents in the consecutive coalition $[1,i]$ upstream of i (including i itself) is equal to $v([1,i])$, while the total payoff to the downstream coalition $[i+1,n]$ is equal to $v(N) - v([1,i]) \geq v([i+1,n])$. In fact, all additional welfare that is realized when the two coalitions $[1,i]$ and $[i+1,n]$ merge to the grand coalition N goes to the downstream coalition. However, any upstream coalition $[1,i]$ can prevent that coalition $[i+1,n]$ gets more than $v([i+1,n])$ by using all its inflows $e_1,\dots,e_i$ by itself. In Herings, van der Laan and Talman (2007) and van den Brink, van der Laan and Vasil'ev (2007) it is alternatively argued that a coalition $[1,i]$ can play some type of ultimatum game by claiming that they will use their total water inflow $\sum_{h=1}^{i} e_i$ by themselves, unless the agents of the downstream coalition $[i+1,j]$ agree to make a monetary compensation almost equal to the total welfare gain of cooperation. This results in precisely the opposite of the solution proposed by Ambec and Sprumont, namely the *upstream incremental welfare distribution*, being the welfare distribution given by[13]

$$f_n^u(v) = v(n) \text{ and } f_i^u(v) = v([i,n]) - v([i+1,n]), \quad i = 2,\dots,n.$$

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[10] In game theory this distribution is known as a marginal vector. Taking any permutation $\pi\colon N \to N$, the *marginal vector* $m^\pi(v) \in \mathbb{R}^n$ of game v induced by π is given by $m_i^\pi(v) = v(\{j \in N | \pi(j) \leq \pi(i)\}) - v(\{j \in N | \pi(j) < \pi(i)\}, i \in N$. The marginal vector assigns to each player its marginal contribution to the coalition of its preceding players in the permutation. In particular, the downstream incremental distribution can be obtained as the marginal vector of the river game, according to permutation $\pi(i) = i$, $i \in N$, i.e., the permutation in which the agents are ordered from upstream to downstream.

[11] A game $v \in \mathcal{G}^N$ is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

[12] Although in deriving the solution core-stableness is only required for the upstream coalitions, it follows that it holds for every coalition.

[13] This distribution also can be obtained as marginal vector of the river game, but now according to permutation $\pi(i) = n-i+1$, $i \in N$, i.e., the permutation in which the players are ordered from downstream to upstream.

In the sequel, we refer to the solution f^u that assigns this welfare distribution to any single spring river game as the *upstream incremental solution*. Although this solution does not satisfy the aspiration level upper bounds, it is also core-stable. For these reasons Herings, van der Laan and Talman (2007) and van den Brink, van der Laan and Vasil'ev (2007) assert that the solution f^u is at least as reasonable as its counterpart f^d.

3.2 Territorial Integration of all Basin States

Another doctrine listed in Kilgour and Dinar (1995, p.1) is: 'The principle of *Territorial Integration of all Basin States*. Symmetrically, this principle favors downstream states, to which it accords 'equal' use, without regard to their contribution to the flow'. This principle is also known as the principle of community (of interests) in the waters, the principle of common management or the drainage basin approach and stated more formally as follows, see e.g. Lipper (1967) or McCaffrey (2001).

Territorial Integration of all Basin States (TIBS): The water of an international watercourse belongs to all basin states combined, no matter where it enters the watercourse. Each basin state is entitled to a reasonable and equitable share in the optimal use of the available water.

The TIBS-principle does not make any country the legal owner of water. Instead, it states that the river water belongs to all the countries combined, no matter where it enters the river, and that each country has the right to 'a reasonable and equitable share' in the optimal (efficient) use of the water. We first notice that TIBS explicitly requires 'efficiency of the water use' and thus requires that the water is allocated in such a way that the total welfare is maximized. Next it is required that each country gets a share in this welfare and thus that the total welfare is distributed amongst the countries. Within the setting of the game-theoretic approach this gives us straightforwardly the following axiom to be satisfied by a solution f on $\mathcal{G}^{(N,\mathcal{U})}$.

Axiom 3.1 Efficiency
A solution f on the class of river games $\mathcal{G}^{(N,\mathcal{U})}$ is **efficient** *if $\sum_{i \in N} f_i(v) = v(N)$ for every $v \in \mathcal{G}^{(N,\mathcal{U})}$.*

Next we elaborate on the TIBS requirement that each country has the right to a 'reasonable and equitable share'. Also with respect to the TIBS principle we have to take into account the one-directionality of the water flow. However notice that, while agent 1 can never obtain more water than enters the river on his own territory, he can reach a higher utility level than the utility he obtains from consuming his own inflow because he has the option to 'trade' some of the water with his downstream neighbor (that possibly derives a higher utility from the water) in exchange for a monetary compensation. The downstream incremental solution for single spring rivers satisfies the TIBS requirement in a particular way. Namely, that every agent is assigned a share in the total welfare equal to his marginal contribution to the welfare that can be obtained by the coalition containing himself and his upstream agents (by allocating their own water inflows optimally amongst themselves). However, as we have seen before, all gains from cooperation with his downstream agents is assigned to the downstream agents. On the other hand, the upstream incremental solution assigns all gains from cooperation between an upstream and downstream coalition to the agents in the former coalition.

In this paper we apply the TIBS requirement that each country should receive a reasonable and equitable share in the maximum welfare, in order to define a new class of solutions, including the above two solutions as extreme cases. To do so, we translate the TIBS principle in a fairness axiom for a solution on the class of TU river games $\mathcal{G}^{(N,\mathcal{U})}$. Suppose that the agents along the river are cooperating according to the coalition structure $\mathcal{P} = \{P^i,\ N_i\}$ for some agent $i \in N \setminus \{n\}$. So, agent i is cooperating with all its upstream agents in coalition P^i, while all other agents are cooperating in its complement coalition N_i. This can happen, for instance, because agent i, for some reason or another, is not willing to send water to its unique downstream neighbor d_i. In this situation the agents in P^i distribute their total welfare $v(P^i)$ and the agents in N_i distribute $v(N_i)$. The question that can then be asked is: how should the gain in welfare $v(N) - v(P^i) - v(N_i)$ that is created by joining the two coalitions be divided among the agents? Evidently, we can ask this question for each agent $i \in N\backslash\{n\}$. The TIBS principle provides us with an answer to this question. Let $\alpha \in \mathbb{R}^n_+$ with $\sum_{h\in N}\ \alpha_h = 1$ be a vector of weights, where $\alpha_h \geq 0$ the weight of agent h, $h \in N$. Then we interpret the TIBS principle by saying that the gain in welfare that is created by joining the two coalitions P^i and N_i, should be divided among these two coalitions proportional to the sum of the weights in these two coalitions. Denoting $\alpha_S = \sum_{i\in S}\ \alpha_i$ for every $S \subseteq N$, we thus require that:

$$\frac{\sum_{k\in P^i} f_k(v) - v(P^i)}{\sum_{k\in N_i} f_k(v) - v(N_i)} = \frac{\alpha_{P^i}}{\alpha_{N_i}},$$

assuming that both α_{P^i} and α_{N_i} are nonzero. This leads to the following fairness axiom (which also is valid in case some weights are zero) for efficient solutions on the class of river games.

Axiom 3.2 α**-TIBS Fairness**
Let $\alpha \in \mathbb{R}^n_+$ *be such that* $\sum_{i\in N} \alpha_i = 1$. *An efficient solution* f *on the class of river games*

$\mathcal{G}^{(N,\mathcal{U})}$ *satisfies* α-**TIBS fairness** *if, for every* $v \in \mathcal{G}^{(N,\mathcal{U})}$ *and* $i \in N\backslash\{n\}$*, it holds that*

$$\alpha_{N_i}\left(\sum_{k\in P^i} f_k(v) - v(P^i)\right) = \alpha_{P^i}\left(\sum_{k\in N_i} f_k(v) - v(N_i)\right). \tag{3.3}$$

Notice that the TIBS principle speaks about agents having the right to 'a reasonable and equitable share' in the optimal use of the water inflow, but does not specify the shares and even does not require the shares to be equal. Therefore we allow for any nonnegative weight vector α which components add up to one. Later we will give special attention to the specific case that all weights are equal.

3.3 Weighted hierarchical solutions

The introduction of efficiency and α-TIBS fairness allows us to find a class of 'fair' (according to these principles) solutions for river games $v \in \mathcal{G}^{(N,\mathcal{U})}$. We call this the class of *weighted hierarchical solutions.* Each solution from this class assigns to every river game $v \in \mathcal{G}^{(N,\mathcal{U})}$ a weighted *hierarchical outcome.* The notion of hierarchical outcome has been introduced by Demange (2004) in the context of games on cycle-free graph structures. In this same context Herings, van der Laan and Talman (2008) propose the average of all hierarchical outcomes as the so-called *average tree solution*, whereas Béal, Rémila and Solal (2010) generalize this solution to the class of all weighted averages of the hierarchical outcomes. In this section we examine the class of weighted hierarchical solutions from the perspective of river games without referring to the underlying graph-theoretical concepts.

For an agent $i \in N$ and an agent k downstream of i, let k^i be the last agent before k on the river branch from i to k (with $k^i = i$ when $k = d_i$). Now take some agent $i \in N$ and for every $k \in N$, consider the following payoff $t^i_k(v)$ (recall from Section 2.2 that Q_i is the set of agents downstream of i, including i itself) given by

$$t^i_k(v) = \begin{cases} v(P^k) - v(P^k \setminus \{k\}) & \text{if } k \in N \setminus Q_i, \\ v(N_{k^i}) - v(N_{k^i}\backslash\{k\}) & \text{if } k \in Q_i\backslash\{i\}, \\ v(N) - v(P^k \setminus \{k\}) - v(N_k) & \text{if } k = i. \end{cases} \tag{3.4}$$

The payoff vector $t^i(v)$ gives a hierarchical outcome of Demange (2004) when i is taken to be 'top' agent in the hierarchy. The set of agents $N \setminus Q_i$ consists of all agents upstream to i and all agents neither upstream nor downstream of i. For instance, in Example 2.1 the set $N \setminus Q_5$ consists of the agents 2, 3 and 4 upstream of 5 and agent 1, which is neither upstream nor downstream of 5. Each agent k not in Q_i receives his marginal contribution to the coalition of agents P^k consisting of this agent k and all his upstream agents. For an agent k downstream of agent i we consider his upstream neighbor k^i on the branch to i. Then such an agent k receives his marginal contribution to the set N_{k^i}, that is the marginal contribution to the set of agents who can be reached from k by walking along the river without visiting his upstream neighbor k^i on the branch from k to i. Notice that $k = d_{k^i}$, i.e., k himself is the (unique) downstream neighbor of k^i. Finally, top agent i receives the surplus $v(N) - v(P^i \setminus \{i\}) - v(N_i)$. Notice that the sets $P^i \setminus \{i\}$ of all agents upstream of i and the set N_i of all agents not in P^i can not cooperate without i and thus $v(N \setminus \{i\}) = v(P^i \setminus \{i\}) + v(N_i)$. So, $t^i_i(v) = v(N) - v(N \setminus \{i\})$, i.e., top agent i receives his marginal contribution to the grand coalition N, which is equal to the additional welfare that he generates by joining together the two coalitions $P^i \setminus \{i\}$ and N_i.

We can consider agent i as the top agent in a hierarchy on the set of all agents as follows. For an agent $h \neq i$, let i_h be the distance from i to h in the river system $(N,\mathcal{U})$, with the distance defined as the number of agents (including h itself, but not i) that has to be visited when going from i to h along the river. For example, taking $i = 4$ in the river system of Figure 1, the distance to agent 3 is one, the distance to agent 2 is two and the distance to agents 1 or 7 is equal to three. Now, let π^i be a permutation on N such that an agent $h \neq i$ is ordered before an agent $k \neq i$ if $i_h > i_k$ (and with agent i as the last agent). So, with $i = 4$ in Figure 1, the agents 1 and 7 with distance three are the two first ordered agents (in arbitrary order), then agents 2 and 6 with distance two, then agents 3 and 5 with distance one and finally agent 4. Then it follows from the fact that for every S the worth $v(S)$ is equal to the sum of the worths of its maximally connected subsets, that $t^i(v)$ is the marginal vector of the game v with respect to this order in which agent i is the last ordered agent. So, considering the river system as a 'hierarchy' with agent i as the top agent, the agents are ordered successively according to their distance to the top and receive their marginal contribution to the coalition of preceding agents. Since the top agent receives everything that is not assigned to the other agents, every hierarchical outcome is efficient and thus the sum of the payoffs is equal to $v(N)$.

Each agent $i \in N$ induces a hierarchical outcome $t^i(v)$, so that the total number of hierarchical outcomes is equal to n. For every nonnegative vector $\alpha \in \mathbb{R}^n_+$, with $\sum_{i \in N} \alpha_i = 1$, the payoff vector $h^\alpha(v) \in \mathbb{R}^n$ given by

$$h^\alpha(v) = \sum_{i \in N} \alpha_i t^i(v) \tag{3.5}$$

is a *weighted hierarchical outcome* of $v \in \mathcal{G}^{(N,\mathcal{U})}$. This gives us the next definition of a weighted hierarchical solution on $\mathcal{G}^{(N,\mathcal{U})}$.

Definition 3.3 *A solution f on the class of river games $\mathcal{G}^{(N,\mathcal{U})}$ is a weighted hierarchical solution if there exists $\alpha \in \mathbb{R}^n_+$ with $\sum_{i \in N} \alpha_i = 1$, such that for every $v \in \mathcal{G}^{(N,\mathcal{U})}$*

$$f_i(v) = h^\alpha_i(v), \ \forall i \in N.$$

We now prove that every weighted hierarchical solution is the unique efficient solution that satisfies the corresponding α-TIBS fairness axiom.

Lemma 3.4 *Given a top agent j, consider the hierarchical outcome $t^j(v)$ and agent i. Then*
(i) $\sum_{k\in P^i} t^j_k(v) = v(N) - v(N_i)$ if $j \in P^i$, and
(ii) $\sum_{k\in P^i} t^j_k(v) = v(P^i)$ if $j \in N_i$.

Proof.
(i) If $j \in P^i$, then for every $h \in N_i$, agent i has to be passed when one goes along the river from j to h. So, for every $h \in N_i$ the distance from j to h is bigger than the distance from j to i, and thus every agent $h \in N_i$ is ordered before i in every permutation π^j that results in the marginal vector $t^j(v)$. Hence all agents of the set N_i are ordered before i and the total payoff $\sum_{h\in N_i} t^j_h(v)$ to these is equal to $v(N_i)$. From efficiency it then follows that the total payment to the agents in the complementary set P^i is equal to $v(N) - v(N_i)$.
(ii) If $j \in N_i$ then $k \in N \setminus Q_j$ for every $k \in P^i$. So every $k \in P^i$ receives $t^j_k(v) = v(P^k) - v(P^k \setminus \{k\})$. Summing up over all $k \in P^i$ gives $\sum_{k\in P^i} t^j_k(v) = v(P^i)$. □

Lemma 3.5 *Let $\alpha \in \mathbb{R}^n_+$ be such that $\sum_{i\in N} \alpha_i = 1$. Then the solution h^α on the class of river games $\mathcal{G}^{(N,\mathcal{U})}$ satisfies α-TIBS fairness.*

Proof. We distinguish three cases.

Case 1. Consider an agent $i \in N$ such that $\alpha_{P^i} > 0$ and $\alpha_{N_i} > 0$. Then

$$\begin{aligned}
\sum_{k\in P^i} h^\alpha_k(v) - v(P^i) &= \sum_{k\in P^i}\sum_{j\in N} \alpha_j t^j_k(v) - v(P^i) \\
&= \sum_{k\in P^i}\sum_{j\in P^i} \alpha_j t^j_k(v) + \sum_{k\in P^i}\sum_{j\in N_i} \alpha_j t^j_k(v) - v(P^i) \\
&= \sum_{j\in P^i}\sum_{k\in P^i} \alpha_j t^j_k(v) + \sum_{j\in N_i}\sum_{k\in P^i} \alpha_j t^j_k(v) - v(P^i) \\
&= \sum_{j\in P^i} \alpha_j \sum_{k\in P^i} t^j_k(v) + \sum_{j\in N_i} \alpha_j \sum_{k\in P^i} t^j_k(v) - v(P^i) \\
&= \alpha_{P^i}(v(N) - v(N_i)) + \alpha_{N_i} v(P^i) - v(P^i) \\
&= \alpha_{P^i}(v(N) - v(N_i) - v(P^i)), \qquad (3.6)
\end{aligned}$$

where the first equality follows by definition of h^α, the fifth equality follows from (i) and (ii) of Lemma 3.4, and the last equality follows since $\alpha_{P^i} + \alpha_{N_i} = 1$.

In a similar way we derive

$$\begin{aligned}
\sum_{k\in N_i} h_k^\alpha(v) - v(N_i) &= \sum_{k\in N_i}\sum_{j\in N}\alpha_j t_k^j(v) - v(N_i)\\
&= \sum_{k\in N_i}\sum_{j\in N_i}\alpha_j t_k^j(v) + \sum_{k\in N_i}\sum_{j\in P^i}\alpha_j t_k^j(v) - v(N_i)\\
&= \sum_{j\in N_i}\sum_{k\in N_i}\alpha_j t_k^j(v) + \sum_{j\in P^i}\sum_{k\in N_i}\alpha_j t_k^j(v) - v(N_i)\\
&= \sum_{j\in N_i}\alpha_j\sum_{k\in N_i} t_k^j(v) + \sum_{j\in P^i}\alpha_j\sum_{k\in N_i} t_k^j(v) - v(N_i)\\
&= \alpha_{N_i}(v(N) - v(P^i)) + \alpha_{P^i}v(N_i) - v(N_i)\\
&= \alpha_{N_i}(v(N) - v(P^i) - v(N_i)). \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7) it follows that

$$\frac{1}{\alpha_{P^i}}\left(\sum_{k\in P^i} h_k^\alpha(v) - v(P^i)\right) = v(N) - v(N_i) - v(P^i) = \frac{1}{\alpha_{N_i}}\left(\sum_{k\in N^i} h_k^\alpha(v) - v(N_i)\right),$$

which shows that the α-TIBS Fairness condition (3.3) is satisfied in this case.

Case 2. Consider an agent $i \in N$ such that $\alpha_{P^i} = 0$. Then $\alpha_{N_i} = 1$ and, since $\alpha_j > 0$ only if $j \in N_i$, we have

$$\sum_{k\in P^i} h_k^\alpha(v) = v(P^i),$$

showing that the α-TIBS Fairness condition (3.3) is also satisfied in this case.

Case 3. Consider an agent $i \in N$ such that $\alpha_{N_i} = 0$. Then, similar as case 2, $\alpha_{P^i} = 1$ and, since $\alpha_j > 0$ only if $j \in P^i$, we have

$$\sum_{k\in N_i} h_k^\alpha(v) = v(N_i),$$

showing that the α-TIBS Fairness condition (3.3) is also satisfied in this case. □

Next we state the characterization result.

Theorem 3.6 *Let $\alpha \in \mathbb{R}_+^n$ be such that $\sum_{i\in N}\alpha_i = 1$. A solution f on the class of river games $\mathcal{G}^{(N,\mathcal{U})}$ satisfies the Efficiency axiom 3.1 and the α-TIBS Fairness axiom 3.2 if and only if it is the weighted hierarchical solution h^α.*

Proof. Since any hierarchical outcome is efficient, also every weighted hierarchical solution is efficient. Further, it follows from Lemma 3.5 that h^α satisfies the α-TIBS Fairness axiom 3.2. It only remains to prove that the two axioms uniquely determine a solution.

Suppose that solution f satisfies the two axioms and let $v \in \mathcal{G}^{(N,\mathcal{U})}$ be a river game. Since equation (3.3) in Axiom 3.2 has to hold for every $i \neq n$, the α-TIBS fairness yields $n-1$ linear independent equations. Together with the Efficiency condition that $\sum_{i \in N} f_i(v) = v(N)$, we thus have n linear independent equations in the n unknown payoffs $f_i(v)$, $i \in N$. Hence the payoffs are uniquely determined and thus it must hold that $f(v) = h^\alpha(v)$, for every $v \in \mathcal{G}^{(N,\mathcal{U})}$. □

We conclude this subsection with considering the core-stability of weighted hierarchical solutions. As mentioned before, in case of a single spring it has been shown by Ambec and Sprumont (2002) that under Assumption 2.2 the river game is convex and thus every marginal vector of a river game v belongs to the core of the game. Hence, in this case every hierarchical outcome is in the core and thus every weighted hierarchical solution, assigning a weighted hierarchical outcome $h^\alpha(v)$ to every single spring river game v, is core-stable. Although, in contrast to the river game with one spring, the river game with multiple springs does not need to be convex, it is superadditive[14] because $v(S)$ is the objective value of the underlying maximization problem. It also holds for every river game with multiple springs $v \in \mathcal{G}^{(N,\mathcal{U})}$ that $v(S \cup T) = v(S) + v(T)$ when $S, T \subset N$, $S \cap T = \emptyset$ and $S \cup T$ is not connected. Then, as has been shown by Demange (2004) within the framework of games on cycle-free graph structures, the core of game v is not empty and every hierarchical outcome satisfies the core lower bounds. Since the core is a convex set, it follows that also every weighted hierarchical outcome is in the core of the game. This yields the following corollary.

Corollary 3.7 *Under Assumption 2.2, every weighted hierarchical solution is core-stable on the class of river games* $\mathcal{G}^{(N,\mathcal{U})}$.

3.4 Average hierarchical solution

When we take $\alpha_n = 1$, and thus $\alpha_i = 0$ for every $i \neq n$, then $h^\alpha(v) = t^n(v)$. Since $j \in P^n$ for every $j \neq n$ and $N_n = \emptyset$, the payoffs of this outcome as given in formula (3.4) reduce to

$$t^n_k(v) = v(P^k) - v(P^k \setminus \{k\}),\ k \in N.$$

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[14] A game $v \in \mathcal{G}^N$ is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for any pair of subsets $S, T \subseteq N$ such that $S \cap T = \emptyset$.

In case of a single spring river game this is the downstream incremental solution as proposed by Ambec and Sprumont. In its general form for games on the class $\mathcal{G}^{(N,\mathcal{U})}$ of river games with (possibly) multiple springs, it is the unique solution characterized by efficiency and the corresponding α-TIBS fairness, which in this specific case of $\alpha_n = 1$ requires that the gain in welfare that is created by joining the two coalitions P^i and N_i, $i \in N$, should be fully allocated to coalition N_i. In other words, joining the coalitions P^i and N_i has no effect on the average utility of the agents in the upstream coalition P^i. According to Corollary 3.7, the generalized downstream incremental solution satisfies the core lower bounds for every river game with multiple springs. In fact, it is straightforward to verify that also in this case it is the unique solution that satisfies the core lower bounds and the Ambec-Sprumont aspiration level upper bounds. The downstream incremental solution for river games with multiple springs was also proposed in Khmelnitskaya (2010) as the unique solution satisfying component efficiency and another property, called successor equivalence, which generalizes the so-called upper equivalence property introduced in van den Brink, van der Laan and Vasil'ev (2007) for line-graph games.

As for the one spring case, this generalized downstream incremental solution for rivers with multiple springs has the disadvantage that all profits of cooperation between an upstream and downstream coalition go to the agents in the downstream coalition, while the agents in the upstream coalition control the water flows from upstream to downstream. In case of one spring, the upstream incremental solution is the weighted hierarchical solution with $\alpha_1 = 1$, thus assigning all weight to the most upstream agent 1. However, in case of multiple springs there is not a unique most upstream agent and therefore there is not a unique straightforward generalization of the upstream incremental solution. One possibility could be to take the average of all hierarchical outcomes corresponding to the agents located on one of the springs, so to take the average over all $t^j(v)$, $j \in O$.

Instead of generalizing the upstream incremental solution, we consider the special case of considering the average of all hierarchical outcomes. For the specific case that all weights are equal, i.e., $\alpha_i = \frac{1}{n}$ for all $i \in N$, α-TIBS fairness yields the following equal weights fairness axiom.

Axiom 3.8 Equal Weights TIBS Fairness
An efficient solution f on the class of river games $\mathcal{G}^{(N,\mathcal{U})}$ satisfies **equal weights TIBS fairness** *if, for every $v \in \mathcal{G}^{(N,\mathcal{U})}$ and any $i \in N \backslash \{n\}$, it holds that*

$$\frac{1}{|P^i|}\left(\sum_{k \in P^i} f_k(v) - v(P^i)\right) = \frac{1}{|N_i|}\left(\sum_{k \in N_i} f_k(v) - v(N_i)\right). \tag{3.8}$$

The unique solution that satisfies Efficiency and the Equal Weights TIBS Fairness axiom is the hierarchical solution corresponding to $\alpha_i = \frac{1}{n}$ for all $i \in N$. In the following, we denote the average hierarchical solution by h^A.[15]

There are several reasons to argue that the average hierarchical solution h^A might be a good alternative for the downstream and upstream incremental solutions. First of all, according to Corollary 3.7 it is core-stable and so for every river game $v \in \mathcal{G}^{(N,\mathcal{U})}$ the core lower bounds reflecting the ATS principle are satisfied by the outcome $h^A(v)$. Second, consider the following. For $S \subseteq N$, let $h^A(S)$ denote the total payoff of the average hierarchical outcome to the agents in S. When, for some $i \neq n$, the upstream coalition P^i is going to cooperate with its complement N_i, then the equal weights TIBS fairness property implies that

$$\frac{1}{|P^i|}\Big(h^A(P^i) - v(P^i)\Big) = \frac{1}{|N_i|}\Big(h^A(N_i) - v(N_i)\Big).$$

Thus, the welfare distribution according to the average hierarchical solution has the property that for every $i \neq n$, the welfare gain of the cooperation of the upstream coalition P^i and its complement coalition N_i is split among the coalition P^i and N_i proportional to the number of agents in these two coalitions, and thus the average welfare gain of an agent in P^i is equal to the average welfare gain of an agent in N_i.

As a special case we consider the implications of this for single spring river games. Recalling that the agents are indexed from upstream to downstream and for $j \geq i$, $[i,j]$ denotes the coalition of consecutive agents $i, i+1, \ldots, j$, the equal weights TIBS fairness property implies that the average weighted hierarchical outcome satisfies for every $k \neq n$

$$\frac{1}{k}\Big(h^A([1,k]) - v([1,k])\Big) = \frac{1}{n-k}\Big(h^A([k+1,n]) - v([k+1,n])\Big).$$

Thus, the welfare distribution according to the average hierarchical solution has the property that for every $k \neq n$, the welfare gain of the cooperation of the upstream coalition $[1,k]$ and the downstream coalition $[k+1,n]$ is split among the coalition $[1,k]$ and $[k+1,n]$ proportional to the number of agents in these two coalitions, and thus the average welfare gain of an agent in $[1,k]$ is equal to the average welfare gain of an agent in $[k+1,n]$. This indeed respects the TIBS principle that, for each k, every agent in the coalition $[k,n]$ is entitled to a share in the optimal use of the water inflow at agent k. For the two agent case it states that the welfare gain of the cooperation between the upstream agent 1 and the

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[15] Within the context of games on cycle-free graph structures this solution is introduced in Herings, van der Laan and Talman (2008) as the Average Tree solution and characterized by the so-called Component Efficiency and Component Fairness axioms. A minor adjustment of these two axioms to the river games setting gives the Efficiency and Equal Weights TIBS Fairness axioms. In Béal, Rémila and Solal (2009) this component fairness axiom is generalized to weighted component fairness for forest games. However, this generalization differs from the α-TIBS fairness, because it assigns weights to so-called cones (of a tree) instead of assigning weights to individual players.

downstream agent 2 is equally distributed between the two agents. It should be noticed that the equal weights TIBS fairness also holds on every subgame. In particular, for some k, let v^k be the subgame on the upstream set of agents $[1,k]$, i.e., this is the k-player TU-game on the set $[1,k]$ when this set does not cooperate with the downstream coalition $[k+1,n]$. Further, let $A^k([i,j])$ denote the total payoff to the agents in the set $[i,j]$, $1 \leq i \leq j \leq k$, when applying the average hierarchical solution on the k-player TU-game v^k. Then the equal weights TIBS fairness property says, for instance, that

$$\frac{1}{k-1}\Big(A^k([1,k-1]) - v([1,k-1])\Big) = A^k([k,k]) - v([k,k]),$$

i.e., when agent k is going to cooperate with its upstream coalition $[1,k-1]$, then the welfare gain to k is a fraction $\frac{1}{k-1}$ of the total welfare gain of the agents in $[1,k-1]$. Analogously, for some k, let v_k be the subgame on the downstream set of agents $[k,n]$, i.e., this is the $n-k+1$-player TU-game on the set $[k,n]$ when this set does not cooperate with the upstream coalition $[1,k-1]$. Further, let $A_k([i,j])$ be the total payoff to the agents in the set $[i,j]$, $k \leq i \leq j \leq n$, when applying the average hierarchical solution on the TU-game v_k. Then the equal weights TIBS fairness property says, for instance, that

$$A_k([k,k]) - v([k,k]) = \frac{1}{n-k}\Big(A_k([k+1,n]) - v([k+1,n])\Big),$$

i.e., when agent k is going to cooperate with its downstream coalition $[k+1,n]$, then the welfare gain to k is a fraction $\frac{1}{n-k}$ of the total welfare gain of the agents in $[k+1,n]$. More general, when either agent $i-1$ or agent $i+k+1$ joins the coalition $[i,i+k]$ of k consecutive agents, then this agent gets a 'fair' share $\frac{1}{k+1}$ of the total welfare gain that results from joining the coalition. In this sense, for single spring river games the average hierarchical solution games meets the symmetric form of the TIBS principle as formulated by Kilgour and Dinar (1995).

4 Weighted hierarchical solutions for river games with externalities

4.1 Rivers with satiable agents

Ambec and Ehlers (2008) have generalized the single spring river game of Ambec and Sprumont by allowing for satiable agents. This means that Assumption 2.2 is weakened by deleting the requirement that the benefit function is strictly increasing.

Assumption 4.1 *In the river game every benefit function* $b^i\colon \mathbb{R}_+ \to \mathbb{R}$ *is a strictly concave function, which is differentiable for* $x_i > 0$ *with derivative going to infinity as* x_i *tends to zero.*

Hence, either b^i is strictly increasing, or there exists a unique number, say s^i, such that b^i is strictly increasing on $x_i < s^i$ and strictly decreasing when $x_i > s^i$. Thus the derivative of b^i is zero at point s^i. The point s^i is the *satiation point* of agent i. We now consider a river with multiple springs and each agent having a benefit function satisfying Assumption 4.1. Without loss of generality (see Ambec and Ehlers, 2008), we further assume that $e_i \leq s^i$ for all i (with $s^i = \infty$ when b^i is strictly increasing).[16]

The existence of satiation points has serious consequences for the resulting game. Before, under Assumption 2.2, only connected coalitions were able to cooperate because any water transfered from one part of a non-connected coalition to another would fully be consumed by 'intermediary' agents. So, a non-connected coalition S consisting of two connected subsets of agents, say an upstream connected subset S_1 and a downstream connected subset S_2, would never transfer water from S_1 to S_2 because the strictly increasing benefit functions of the agents would make that all water sent from S_1 to S_2 would immediately be taken by the agents in-between S_1 and S_2. In contrast, under Assumption 4.1 it might be profitable for a non-connected coalition to transfer water between its non-connected subsets. When all agents in-between S_1 and S_2 have a satiation point then it might be profitable for the not connected coalition $S = S_1 \cup S_2$ to send water from its upstream part S_1 to its downstream part S_2. Although some of this flow might be taken by the in-between agents, these agents will only take water up to their satiation points. So, when the flow is big enough, part of it will reach S_2, possibly rendering cooperation between the two non-connected parts of the coalition profitable. As a result, the worth of a non-connected coalition S can now be higher than the sum of the worths of its maximal connected subsets.

This phenomenon might cause positive *externalities* on a connected coalition T. Under Assumption 2.2, the worth of T follows from the maximization program (2.2). However, when all agents in T have satiation points, and it is profitable for agents upstream of T to send water to agents downstream of T, the worth of T depends on the coalition formation of the agents outside T. Situations in which the worth of a coalition $S \subset N$ depends on the coalition formation outside S can be modeled by a *cooperative game with transferable utility in partition function form*, or shortly a PFF-game, as proposed in Thrall and Lucas (1963).[17] A partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of the set N into k subsets denotes a cooperation structure within the grand coalition N, i.e., players in set T cooperate if and only if $T \in \mathcal{P}$. Then a PFF-game assigns a worth $v(S, \mathcal{P})$ to for every pair $(S, \mathcal{P})$ such that $S \in \mathcal{P}$, i.e.,

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[16] If $e_i > s_i$ then in the optimal allocation problem $e_i - s_i$ can be considered as additional inflow at its downstream neighbor d_i.

[17] Despite its early introduction, the partition function form turned out to be a methodological tough problem, and only during the last decade breakthroughs are reported, see e.g. Funaki and Yamato (1999), Ray and Vohra (1999), Albizuri, Arin and Rubio (2005), Gomez (2005), Macho-Stadler, Pérez-Castrillo and Wettstein (2007), De Clippel and Serrano (2008) and Dutta, Ehlers and Kar (2010).

the worth of a coalition S in $\mathcal{P}$ of N depends on the cooperation structure $\mathcal{P} \setminus \{S\}$ of the players in $N \setminus S$. For $S \in \mathcal{P}$, the worth $v(S, \mathcal{P})$ denotes the maximum welfare (sum of the benefits) that the players in S can guarantee themselves by cooperating, when the players outside S form coalitions T, $T \in \mathcal{P} \setminus \{S\}$.[18] When we take the river system $(N, \mathcal{U})$ as given, we denote the collection of all partition function form games v on $(N, \mathcal{U})$ with benefit functions satisfying Assumption 4.1 by $\mathcal{PG}^{(N,\mathcal{U})}$.

When $S \in \mathcal{P}$ and every coalition $T \in \mathcal{P}$, $T \neq S$, is a singleton, then the agents outside S act individually and every $i \notin S$ consumes at least its own water inflow e_i (because $e_i \leq s^i$ for all i). We denote $\mathcal{P}_S = \{S\} \cup \{\{i\}, i \notin S\}$ as the partition where all agents outside S do not cooperate and act as singletons, and $v_*(S) = v(S, \mathcal{P}_S)$ as the worth of S in the corresponding partition function form game. This worth is called (Ambec and Ehlers, 2008) the (non-cooperative) core lower bound of S. For a connected coalition we have that $v_*(S) = v(S)$, i.e., $v_*(S)$ is precisely the worth that S can obtain by solving the welfare maximizing problem (2.2), and thus is equal to the welfare that the agents in S obtain when allocating optimally their own water inflows amongst themselves. We further denote $v^*(S) = v(S, \mathcal{P}^S)$, with $\mathcal{P}^S = \{S,\ N \setminus S\}$ the partition such that all agents not in S work together. The worth $v^*(S)$ is called the (cooperative) core upper bound of S. It is the amount that the agents in S can guarantee themselves when the agents outside S cooperate together in coalition $N \setminus S$. Notice that $v_*(S)$ and $v^*(S)$ are defined for every S, not only for connected coalitions. Also notice that $v_*(N) = v^*(N) = v(N, \{N\}) = v(N)$ is the worth of the grand coalition when all agents cooperate together.

The following results have been stated in Ambec and Ehlers (2008) for rivers with one spring and generalize straightforwardly to rivers with multiple springs.

Lemma 4.2 *Let $\mathcal{P}$ be a partition of N. Then*
(i) $v_*(S) \leq v(S, \mathcal{P})$, for all $S \in \mathcal{P}$.
(ii) For any two different $S, T \in \mathcal{P}$, $v(S, \mathcal{P}) + v(T, \mathcal{P}) \leq v(S \cup T, \mathcal{P}')$ with $\mathcal{P}' = (\mathcal{P} \setminus \{S, T\}) \cup \{S \cup T\}$.

Notice that (i) also implies that $v_*(S) \leq v^*(S)$ and that (ii) implies that

$$v_*(S) + v_*(T) \leq v_*(S \cup T)$$

for every disjoint S and T. Hence, the worths $v_*(S)$, $S \subseteq N$, induce a superadditive TU-game (N, v_*). Recall that in the case without externalities the above inequality holds with

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[18] For a river with one spring, Ambec and Ehlers (2008) provides an iterative procedure to find the worths $v(S, \mathcal{P})$ for every $\mathcal{P}$ and every $S \in \mathcal{P}$. For general rivers, it is an open question whether $v(S, \mathcal{P})$ is uniquely determined and if so, how to derive the value. However, we will show that our solutions only depend on $v(S, \mathcal{P})$ for $S = P^k$ or $S = N_k$, $k \in N$. For these sets S the values do not depend on the partition $\mathcal{P}$ (see Theorem 4.3) and follow from solving the maximization problems (2.2).

equality when S and T are two disjoint connected coalitions and $S \cup T$ is not connected. This does not need to be true in the river game with satiation points. For instance when S is upstream of T, the union $S \cup T$ may benefit from a water flow going from S to T. For rivers with a single spring it is also stated in Ambec and Ehlers (2008) that for coalitions $S = [1, i] = \{1, \ldots, i\}$, $i = 1, \ldots, n$,

$$v([1, i], \mathcal{P}) = v_*([1, i]) \text{ for every } \mathcal{P} \text{ with } [1, i] \in \mathcal{P},$$

i.e., when coalition S consists of some agent i and all its upstream agents, then the worth of S does not depend on the partition of the agents outside S. Indeed, by definition, the worth of such an upstream coalition S does not depend on the behavior of the agents downstream of S. In case of a river with multiple springs, this result generalizes to every upstream coalition P^k and its complement $N_k = N \setminus P^k$, $k \in N$.

Theorem 4.3 *Let $v \in \mathcal{PG}^{(N,\mathcal{U})}$ be a partition function form river game on $(N,\mathcal{U})$. Then, for every partition $\mathcal{P}$ and $S \in \mathcal{P}$,*

$$v(S, \mathcal{P}) = v_*(S) \textit{ if } S = P^k \textit{ or } S = N_k, \quad \textit{for some } k \in N.$$

Proof. Let $S = P^k$ for some $k \in N$. Since P^k consists of agent k and all its upstream agents, its worth $v(P^k, \mathcal{P})$ does not depend on the partition $\mathcal{P} \setminus \{P^k\}$ of the agents outside P^k. Hence $v(P^k, \mathcal{P}) = v_*(P^k)$ for all $\mathcal{P}$ with $P^k \in \mathcal{P}$.

Next consider $S = N_k$ for some $k \in N$. By definition of P^k, agent k is the only agent in P^k that is connected to an agent in N_k, namely to its unique downstream neighbor d_k. Further, by definition of P^k, there are no agents in P^k downstream of k. By the assumption that $e_h < s^h$ for every $h \in N$, it follows that agent d_k never receives any water from k, independent of the partition of $P^k = N \setminus N_k$. So, $v(N_k, \mathcal{P}) = v_*(N_k)$ for all $\mathcal{P}$ with $N_k \in \mathcal{P}$. $\square$

We say that the worth of every coalition of type P^k or type N_k is *externality-free*, i.e., the worth does not depend on the partition of the agents outside a coalition of such a type.

4.2 Solutions for rivers with satiable agents

Next, we consider the application of weighted hierarchical solutions to the class $\mathcal{PG}^{(N,\mathcal{U})}$ of partition function form river games with externalities. Similar as without externalities, we speak about an efficient solution if it always allocates the worth of the grand coalition N when all agents cooperate, thus the worth $v(N) = v(N, \{N\})$ with $v(N)$ the solution of maximization problem 2.2 for $S = N$.

Axiom 4.4 Efficiency for river games with externalities
A solution f on the class of river games with externalities $\mathcal{PG}^{(N,\mathcal{U})}$ is **efficient** *if it holds for any game $v \in \mathcal{PG}^{(N,\mathcal{U})}$ that $\sum_{i\in N} f_i(v) = v(N, \{N\})$.*

Also the TIBS principle can be applied to river games with externalities in a similar way as for the no-externality case. Let $\mathcal{P}(i)$ denote the coalition structure $\{P^i,\ N_i\}$ and recall that the TIBS fairness axiom for games without externalities was obtained by considering the situation that the agents along the river are cooperating according to the coalition structure $\mathcal{P}(i)$ for some agent $i \neq n$, i.e., agent i is cooperating with all its upstream agents in coalition P^i, while all other agents are cooperating in its complement coalition N_i. Under externalities, the agents in P^i can obtain total welfare $v(P^i, \mathcal{P}(i))$ and the agents in N_i earn $v(N_i, \mathcal{P}(i))$. However, recall from Theorem 4.3 that these worths of coalitions of type P^i and N_i are externality-free and thus $v(P^i, \mathcal{P}(i)) = v_*(P^i)$, respectively $v(N_i, \mathcal{P}(i)) = v_*(N_i)$, being the non-cooperative core lower bounds of these coalitions. This gives us the next α-TIBS fairness axiom for the class of river games with externalities.

Axiom 4.5 α-TIBS Fairness for river games with externalities
*Let $\alpha \in \mathbb{R}^n_+$ be such that $\sum_{i\in N} \alpha_i = 1$. An efficient solution f on the class of river games with externalities $\mathcal{PG}^{(N,\mathcal{U})}$ satisfies α-***TIBS fairness** *if, for any $v \in \mathcal{PG}^{(N,\mathcal{U})}$ and any $i \in N\backslash\{n\}$, it holds that*

$$\alpha_{N_i}\left(\sum_{k\in P^i} f_k(v) - v_*(P^i)\right) = \alpha_{P^i}\left(\sum_{k\in N_i} f_k(v) - v_*(N_i)\right). \tag{4.9}$$

Notice that this is the same as for the no externality case, only the worths of the coalitions P^i and N_i are replaced by their non-cooperative core lower bound values in the partition function form game v. So, irrespective of externalities, this axiom states that the gain in welfare that is created by joining the two coalitions P^i and N_i, should be divided among the two coalitions proportional to the sum of the weights in these two coalitions.

Similar as in the proof of Theorem 3.6 it follows that there is a unique solution that satisfies efficiency and α-TIBS fairness for $\alpha \in \mathbb{R}^n_+$ with $\sum_{i\in N} \alpha_i = 1$. Moreover, similar as in the proof of Lemma 3.5 it follows straightforwardly that the weighted hierarchical solution to the associated non-cooperative lower bound TU-game v_* satisfies both axioms. This shows the following theorem.

Theorem 4.6 *Let $\alpha \in \mathbb{R}^n_+$ be such that $\sum_{i\in N} \alpha_i = 1$. A solution f on the class of river games with externalities $\mathcal{PG}^{(N,\mathcal{U})}$ satisfies Efficiency and α-TIBS Fairness if and only if $f(v) = h^\alpha(v_*)$ for every $v \in \mathcal{PG}^{(N,\mathcal{U})}$.*

As before, also on the class of river games with multiple springs and externalities we refer to solutions as characterized in this theorem as weighted hierarchical solutions. It assigns for a given weight vector α with components adding up to one, the corresponding weighted hierarchical outcome of the associated TU-game v_*. We next show that every weighted hierarchical solution is *externality-free.*

Definition 4.7 *A solution f on the class of river games with externalities $\mathcal{PG}^{(N,\mathcal{U})}$ is externality-free if the payoffs only depend on the worths of the externality-free coalitions.*

We first consider the hierarchical outcome $t^i(v_*)$ in case the river has only one spring. Then formula (3.4) reduces to

$$t^i_k(v_*) = \begin{cases} v_*([1,k]) - v_*([1,k-1]) & \text{if } k < i, \\ v_*(N) - v_*([1,k-1]) - v_*([k+1,n]) & \text{if } k = i, \\ v_*([k,n]) - v_*([k+1,n]) & \text{if } k > i. \end{cases} \tag{4.10}$$

An agent upstream of agent i receives its marginal contribution to the coalition of agents consisting of this agent and all agents upstream to it. An agent downstream of agent i receives its marginal contribution to the coalition of agents consisting of this agent and all agents downstream to it. Finally, agent i receives its marginal contribution to the grand coalition N, i.e., agent i receives the benefit of cooperation that is obtained by connecting the upstream coalition $[1, i-1]$ and the downstream coalition $[i+1, n]$. Further, formula (4.10) shows that in every hierarchical outcome the payoffs are fully determined by the worths $v(S)$ with S of either type $[1, j]$ or type $[j, n]$ for some $1 \leq j \leq n$, i.e., the payoffs are fully determined by the worths of the upstream coalitions $[1, j]$ and the downstream coalitions $[j, n]$, $j = 1, \ldots, n$. The worths of all other coalitions $[i, j]$, $1 < i < j < n$ of consecutive agents don't affect the payoffs. Noticing that, for $i < n$, $P^i = [1, i]$ and $N_i = [i+1, n]$ it follows from Theorem 4.3 that every coalition that appears in the formula above is externality-free, illustrating that in case of a river with only one spring, the hierarchical outcome only depends on the worths of the externality-free coalitions and so every weighted hierarchical solution is externality-free for rivers with only one spring.

We now consider the case of multiple springs. For some i, first consider an agent $k \in N \setminus Q_i$. Then, according to formula (3.4) the payoff to k in $t^i(v_*)$ is given by

$$t^i_k(v_*) = v_*(P^k) - v_*(P^k \setminus \{k\}).$$

Since $v_*(P^k \setminus \{k\}) = \sum_{j \in U^k} v_*(P^j)$ it follows that the payoff $t^i_k(v_*)$ only depends on the worths of coalitions of type P^j, $j \in N$. Second, consider an agent $k \in Q_i \setminus \{i\}$. Then, according to formula (3.4) the payoff to k in $t^i(v_*)$ is given by

$$t^i_k(v_*) = v_*(N_{k^i}) - v_*(N_{k^i} \backslash \{k\}).$$

Inspecting $v_*(N_{k^i} \backslash \{k\})$ it follows that $v_*(N_{k^i} \backslash \{k\}) = v_*(N_k) + \sum_{h \in U^k \backslash \{k^i\}} v_*(P^h)$. Hence, every term in the expression above is either of type $v_*(P^i)$ for some $i \in N$ or of type $v_*(N_i)$ for some $i \in N$. Apparently, also the payoff to player i only depends on the worths of coalitions of type P^j and N_j, $j \in N$. Since, according to Theorem 4.3 every coalition of these types is externality-free, it follows that every hierarchical outcome $t^i(v_*)$ only depends on the worths of the externality-free coalitions. This gives the following two corollaries.

Corollary 4.8 *On the class $\mathcal{PG}^{(N,\mathcal{U})}$ of river games with externalities, every weighted hierarchical solution h^α assigning payoff $h^\alpha(v_*)$ for every $v \in \mathcal{PG}^{(N,\mathcal{U})}$ is externality-free.*

Corollary 4.9 *On the class $\mathcal{PG}^{(N,\mathcal{U})}$ of river games with externalities, the axioms of Efficiency and α-TIBS Fairness imply externality-freeness.*

We now consider core stability. Every weighted hierarchical outcome is a convex combination of the hierarchical outcomes $t^i(v_*)$, $i \in N$. As seen before, for a river game $v \in \mathcal{G}^{N,\mathcal{U})}$ without externalities, every hierarchical outcome is in the core of the game v. However, as we have seen in Section 3.3, for two connected, disjoint coalitions S and T the worth $v_*(S \cup T)$ can be bigger than the sum of the two worths $v_*(S)$ and $v_*(T)$. Thus, a hierarchical outcome $t^i(v_*)$ does not need to satisfy the non-cooperative core lower bound $v_*(R)$ for every coalition R. However, in Ambec and Ehlers (2008) it is argued that in river games it is natural to restrict blocking for connected coalitions, because coordination among agents becomes difficult when the agents are not neighboring. Clearly, every hierarchical outcome satisfies the non-cooperative core lower bound for every connected coalition R. Hence, we have the following corollary.

Corollary 4.10 *For a river game with externalities $v \in \mathcal{PG}^{(N,\mathcal{U})}$, a weighted hierarchical solution satisfies the non-cooperative core lower bounds when blocking is restricted to connected coalitions.*

It should be noticed that it is shown in Ambec and Ehlers (2008) that for the river game with a single spring and satiable agents, the downstream incremental solution (i.e. the hierarchical outcome $t^n(v_*)$) satisfies all non-cooperative core lower bounds. It is an open question whether this also holds for river games with multiple springs.

Finally, we would like to mention that also on the class of river games with multiple springs and externalities the average hierarchical solution satisfies Equal Weights TIBS-fairness. Hence, when an agent k stops cooperating with its unique downstream neighbor such that coalition P^k consisting of k and all its upstream neighbors does not want to cooperate with its complement N_k, then the total payoff to the agents in P^k is equal to $v_*(P^k)$ and the total payoff of the agents in N_k is equal to $v_*(N_k)$. When the two coalitions cooperate and distribute the total payoff according to the average hierarchical solution, also in the game with externalities the average welfare gain of the agents in P_k is equal to the average welfare gain of the agents in N_k.

5 Concluding remarks

In this paper we applied principles for water allocation to define a class of solutions for allocating river water among the agents located along the river. In particular, we utilized the principle of Territorial Integration of all Basin States (TIBS principle), which requires that (i) the water is assigned in such a way that the total welfare of all countries is maximized (optimal use), and (ii) each country gets a (reasonable and equitable) share in the total welfare resulting from an optimal assignment. The first part is applied as the well-known efficiency requirement. The second part is translated by assigning to every agent a nonnegative weight, such that these weights add up to 1, and requiring that whenever an agent (except the most downstream agent) is not cooperating with its downstream neighbor, then the resulting loss in welfare is distributed among the two resulting sub-river systems proportional to the sum of the weights of the agents in these sub-river systems.

The solutions that are obtained in this way are the weighted hierarchical solutions, with the downstream incremental and upstream incremental solutions as extreme cases, and the average hierarchical solution as some 'compromise' where all agents have an equal weight. We applied these principles to rivers with and without satiable agents (yielding externalities when non-connected coalitions cooperate), and (possibly) multiple springs.

Besides the theoretical insight obtained by the axiomatizations of the new class of solutions, the axiom of α-weighted fairness is useful to make the TIBS principle 'operational' for water allocation. First, it gives a precise formulation of the TIBS principle that, as we showed, can be used to define water allocation solutions. Accepting this interpretation of the TIBS principle, we only need to determine the α-weights of the agents. When these weights are exogenously given (for example by existing power structures among countries), they can be applied directly. In case the weights are not given, they can be the subject of negotiation. An advantage is that the weights are assigned to agents, and not to coalitions (in fact the weight of a coalition is just the sum of the weights of the agents in the coalition), so it is not necessary to negotiate for each possible split of the river system to determine the weights of the resulting sub-river systems.

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