An Ascending Multi-Item Auction with Financially Constrained Bidders*

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Abstract

A number of heterogeneous items are to be sold to a group of potential bidders. Every bidder knows his own values over the items and his own budget privately. Due to budget constraint, bidders may not be able to pay up to their values. In such a market, a Walrasian equilibrium usually fails to exist and also the existing auctions might fail to allocate the items among the bidders. In this paper we first introduce a rationed equilibrium for a market situation with financially constrained bidders. Succeedingly we propose an ascending auction mechanism that always results in an equilibrium allocation and price system. By starting with the reservation price of each item, the auctioneer announces the current prices of the items in each step and the bidders respond with their demand sets at these prices. As long as there is overdemand, the auctioneer adjusts prices upwards for overdemanded items until a price system is reached at which either there is an underdemanded set, or there is neither overdemand nor underdemand anymore. In the latter case the auction stops. In the former case, precisely one item will be sold, the bidder buying the item leaves the auction and the auction continues with the remaining items and the remaining bidders. We prove that the auction finds a rationed equilibrium in a finite number of steps. In addition, we derive various properties of the allocation and price system obtained by the auction.

Keywords: Ascending auction, multi-item auction, financial constraint.

JEL classification: D44.

1 Introduction

Auctions are typically the most efficient institution for the allocation of private goods and have been long used since antiquity for the sale of a variety of items. The academic study of auctions grew out of the pioneering work of Vickrey (1961) and has blossomed into an enormously important area of economic research over the last 30 years. The development in the area has been further accelerated as today governments are keen on using auctions to sell spectrum rights, to procure goods and service, and to privatize state enterprises. Also consumer-oriented online auctions are booming to sell virtually all sorts of commodities. The research of the last four decades resulted in a better understanding of how the design of auction affects its outcome and how the environments may affect the design of auctions as well.

Standard auction theory assumes that all potential bidders have the ability to pay up to their values on the items for sale. However in reality many buyers may be financially constrained and may therefore not be able to afford what the items are worth to them. Financial or budget constraints may occur in various circumstances. As stressed by Maskin (2000) in his Marshall lecture, the consideration of financial constraints on buyers is particularly relevant and important in many developing countries, where auctions are used to privatize state assets for the promotion of efficiency, competition and development, but entrepreneurs may often be financially constrained. Financial constraints not only occur in developing countries but also in developed nations. In particular, Che and Gale (1998) have given a variety of situations where financial constraints may arise, ranging from an agent's moral hazard problem and business downturns, to the acquisition decisions in many organizations which delegate to their purchasing units but impose budget constraints to control their spending, and to the case of salary caps in many professions where budget constraints are used to relax competition.

Financial constraints can pose a serious obstacle to the efficient allocation of the items. For instance, financial constraints seem to have played an important role in the outcome of auctions for selling spectrum licenses conducted in US (see McMillan (1994) and Salant (1997)) and in European countries (see Illing and Klüh (2003)). In this paper, we study a general model in which a number of (indivisible) items are sold to a group of financially constrained bidders. Each bidder wants to consume at most one item. When no bidder faces a financial constraint, the model reduces to the well-known assignment model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981), and Demange, Gale and Sotomayor (1986), among others. Each bidder has private information about his values for the items and his budget, but these numbers are not revealed to the other agents. In particular the auctioneer (seller) does not know the values and budgets of the bidders and her own reservation prices is also private information and is not revealed to the bidders. It is well-known (see e.g., Maskin (2000)) that even when a single item is auctioned, it is generally impossible to have a mechanism for achieving the full market efficiency in case bidders face budget constraints. Of course, this observation also holds when there are multiple items for sale. Even worse, in situations that bidders face financial constraints, a Walrasian equilibrium typically fails to exist and also allocation mechanisms that result in an efficient allocation when bidders are unconstrained might even fail to find a feasible allocation.

The natural question therefore is whether for situations with financial constraints a market mechanism can be designed that at least always results in a feasible outcome and might arrive at an outcome as efficient as possible. In this paper we first define a rationed equilibrium for a market situation with financially constrained bidders. Succeedingly we propose an ascending auction mechanism that always results in an equilibrium allocation and price system. The auction can be seen as a modification of the well-known ascending auction of Demange, Gale and Sotomayor (1986) for situations without financial constraints. The auctioneer starts with the seller's reservation price vector, that specifies the lowest admissible price for each item, and each bidder responds with a set of items demanded at those prices. The auctioneer adjusts prices upwards for overdemanded items until either a price system is reached for which a set of items is underdemanded or a price system with no underdemand and no overdemand at all. In the first case precisely one item is assigned to a bidder that demanded that item at the previous price system, while other bidders that demanded the item are excluded from the market for that item and thus effectively are rationed on their demand for this item. Such a rationing can only occur when there are several bidders with a budget equal to the selling price of the item. In case there is neither underdemand nor overdemand anymore, an equilibrium has been reached for all the remaining items. We prove that the auction finds a rationed equilibrium (and thus feasible allocation) in a finite number of steps. An attractive feature of the auction is that it only requires the bidders to report their demands at several price vectors along a finite path rather than their values or budgets. This property is very useful and practical, because agents generally refuse to reveal their values or budgets. This is also a reason why English and Dutch auctions are much more popular than sealed-bid auctions. We also derive various properties of the auction and we discuss on its efficiency. In particular, we demonstrate that when there is no budget constraint for any bidder, the new auction coincides with the auction of Demange, Gale and Sotomayor (1986).

We conclude this introduction by reviewing related literature. Rothkopf (1977) was among the first to study some issues concerning sealed-bid auctions with budget constrained bidders. He investigated how such constraints may affect the best bids of a bidder. Palfrey (1980) analyzed a price discriminatory sealed-bid auction in a multiple item setting under budget constraints and gave a complete characterization of Nash equilibrium in the two items or less and two bidders or less case. Pitchik and Schotter (1988) studied the equilibrium bidding behavior in sequential auctions for the sale of two items with budget constrained bidders. Che and Gale (1996, 1998) focused on single item auctions with budget constraints under incomplete information. They proved that when bidders are subject to financial constraints, the well-known revenue equivalence theorem does not hold any more. In particular, Che and Gale (1998) provided conditions under which first-price auctions yield higher expected revenue and social surplus than second-price auctions; see also Krishna (2002) and Klemperer (2004). Laffont and Robert (1996) characterized an optimal sealed-bid auction in a single item setting under financial constraints. Maskin (2000) studied the performance of second-price auctions and all-pay auctions and proposed a constrained-efficient sealed-bid auction for the sale of a single item when bidders are financially constrained. Finally, Benoît and Krishna (2001) investigated simultaneous ascending auctions and sequential auctions for the sale of two items with budget constrained bidders. They compared the performance of both types of auctions when the two items are complements or substitutes; see also Krishna (2002).

This paper is organized as follows. Section 2 presents the model and the notion of rationed equilibrium. Section 3 derives a number of basic properties of over and under demanded sets. Section 4 introduces the auction mechanism and proves its convergence. In Section 5 it is shown that the auction results in a rationed equilibrium. Also some other the properties of the allocation and prices generated by the auction are discussed. Section 6 concludes.

2 The Model

A seller or auctioneer has n indivisible goods for sale to a set of m financially constrained bidders. Let $N = \{1, \ldots, n\}$ denote the set of the items for sale and $M = \{1, 2, \cdots, m\}$ the set of bidders. In addition to the n real items there is a *dummy good*, denoted by 0. The dummy item 0 can be assigned to any number of bidders simultaneously, any real item $j \in N$ can be assigned to at most one bidder. Every bidder (he) wants to buy at most one good. The seller (she) has for each real item $j \in N$ a nonnegative integer reservation price $c(j)$ below which the item will not be sold. The seller will not reveal these values $c(j)$ until the auction starts. By convention, the reservation price of the dummy good is known to be $c(0) = 0$.

Each bidder i is initially endowed with a nonnegative integer amount of m^i units of money. Further, every bidder $i \in M$ attaches a (possibly negative) integer monetary value to each item in $N \cup \{0\}$ given by the valuation function $V^i \colon N \cup \{0\} \to \mathbb{Z}$. Also by convention, the value of the dummy item for every buyer i is known to be $V^i(0) = 0$. The value $V^i(j)$ of each real item $j \in N$ and the amount m^i are private information and thus only known by bidder i himself. Buying an item j against price p_j by bidder i yields him a utility U^i equal to

$$U^i = \begin{cases} V^i(j) + m^i - p_j & \text{if } p_j \leq m^i, \\ -\infty & \text{if } p_j > m^i. \end{cases}$$

That is, no bidder can afford a price above his budget m^i. Since no bidder i will ever pay more than $V^i(j)$ for any item j, his financial constraint m^i is never binding when $m^i \geq \max_{j \in N} V^i(j)$. In this case we have precisely the well known assignment market model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981), and Demange, Gale and Sotomayor (1986). In this paper we allow for the possibility that there exist bidders i with $m^i < \max_{j \in N} V^i(j)$, i.e., there might exist bidders whose value on some items exceeds what they can afford.

A *feasible allocation* π assigns every bidder $i \in M$ precisely one item $\pi(i) \in N \cup \{0\}$ such that no real item $j \in N$ is assigned to more than one bidder. Note that a feasible allocation may assign the dummy good to several bidders and that a real item $j \in N$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let N_π denote the set of unassigned items at π, i.e., $j \in N_\pi$ if $j \neq \pi(i)$ for all $i \in M$. A feasible allocation π^* is *socially efficient* if

$$\sum_{i \in M} V^i(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \geq \sum_{i \in M} V^i(\pi(i)) + \sum_{j \in N_\pi} c(j)$$

for every feasible allocation π, so a socially efficient allocation maximizes the total value that can be obtained from allocating the items over all agents. A price vector $p \in \mathbb{R}^{n+1}$ yielding a price p_j for every item $j \in N \cup \{0\}$ is *feasible* if $p_j \geq c(j)$ for every $j \in N$ and $p_0 = 0$. A pair (p, π) of a feasible price vector p and a feasible allocation π is said to be *implementable* if $p_{\pi(i)} \leq m^i$ for all $i \in M$, i.e., every bidder i can afford to buy the item $\pi(i)$ assigned to him. Given a feasible price vector $p \in \mathbb{R}^{n+1}$, the demand set of bidder i is defined by

$$D^i(p) = \{j \in N \cup \{0\} \mid p_j \leq m^i,\ V^i(j) - p_j = \max_{\{k \in N \cup \{0\} \mid p_k \leq m^i\}} (V^i(k) - p_k)\}.$$

So an item $j \in N \cup \{0\}$ is in the demand set $D^i(p)$ if and only if it can be afforded at p and it maximizes the surplus $V^i(k) - p_k$ over all affordable items k. Observe that for any feasible p, the demand set $D^i(p) \neq \emptyset$, because $p_0 = 0 \leq m^i$ and thus the dummy item is always affordable.

Definition 2.1 *A Walrasian equilibrium* (**WE**) *is an implementable pair* (p, π) *such that*
(a) $\pi(i) \in D^i(p)$ *for all* $i \in M$,
(b) $p_j = c(j)$ *for any unassigned item* $j \in N_\pi$.

If (p, π) is a WE, then p is called an equilibrium price vector and π an equilibrium allocation. From Shapley and Shubik (1972) it is well known that in a situation without financial constraints a Walrasian equilibrium exists and that any equilibrium equilibrium allocation is socially efficient and therefore also Pareto efficient. To find a Walrasian equilibrium in a situation without financial constraints one can apply the auction mechanism of Demange, Gale and Sotomayor (1986), see also Crawford and Knoer (1981), using the notion of overdemanded set. A subset of N is *overdemanded* at a price vector p if the number of buyers who demand goods only from this set is greater than the number of items in that set. Demange *et al.* propose an ascending auction in which the auctioneer starts with the reservation price vector given by $p_0 = 0$ and $p_j = c(j)$, $j \in N$. Then each bidder is required to respond with his demand set $D^i(p)$ of most prefered items at price p. When there is an overdemanded set of goods, the price of any item j in a *minimal* overdemanded set (i.e., no strict subset of this overdemanded set is overdemanded) is increased by one and the bidders have to resubmit their demands at this new price vector. The auction stops as soon as there are no overdemanded sets anymore. It is well-known that the auction stops in a finite number of steps with a minimal equilibrium price vector $p^{\min}$, i.e., (i) there exists a feasible allocation π^* such that (p^{min}, π^*) constitutes a Walrasian equilibrium and (ii) it holds that $p \geq p^{min}$ for any other equilibrium price p. Note that in the single item case, the auction of Demange *et al.* (the DGS auction in short) reduces to the English auction. These results, however, do not hold in the case of financial constraints. Namely, the existence of an equilibrium cannot be assured anymore and also the DGS auction might fail when some bidders face financial constraints, as illustrated in the next simple example with only one item.

Example 1. One real item is auctioned amongst two bidders. Suppose that $c(1) = 0$, $V^1(1) = 5$ and $V^2(1) = 6$. We consider three situations with respect to the budget constraints.

Case 1: $m^1 = m^2 > 6$. In this case the financial constraints are never binding. At the starting price vector $p = (p_0, p_1)^\top = (0, 0)^\top$, we have that both bidders demand the real item 1 and $N = \{1\}$ is a minimal overdemanded set. This is also the case for any $p_1 < 5$. When $p_1 = 5$, then $D^1(p) = \{0, 1\}$ (bidder one is indifferent between the real item 1 and the dummy item 0) and $D^2(p) = \{1\}$. Thus N is not overdemanded anymore, because at this price bidder 1 also has the dummy item in his demand set and so the number of bidders that only have items from N is equal to one. The equilibrium price is $p^* = (0, 5)$ with allocation $\pi^*(1) = 0$ and $\pi^*(2) = 1$. Observe that any $p = (0, p_1)^\top$ with $5 \leq p_1 \leq 6$ is an equilibrium price, so the ascending auction (i.e., English auction) ends up with the minimal equilibrium price. Also observe that the equilibrium allocation is socially efficient.

Case 2: $m^1 = 3$ and $m^2 = 2$. In this case we have that at $p = (0, 2)^\top$ the demands are given by $D^1(p) = D^2(p) = \{1\}$, and at $p = (0, 3)^\top$ by $D^1(p) = \{1\}$ and $D^2(p) = \{0\}$. So, the latter price system yields an equilibrium in which the item is allocated to bidder 1 against price $p_1 = 3$. Observe that the equilibrium allocation is not socially efficient. Of course, when $m^1 = 2$ and $m^2 = 3$, the auction assigns the item to bidder 2 at price 3 and the outcome is socially efficient.

Case 3: $m^1 = m^2 = 2$. We now have that at $p = (0, 2)^\top$ the demands are given by $D^1(p) = D^2(p) = \{1\}$, and at $p = (0, 3)^\top$ by $D^1(p) = D^2(p) = \{0\}$. Now an equilibrium does not exist and also the DGS auction fails to allocate the item to one of the bidders, although both bidders have valuations and budgets above the zero reservation price of the seller. At $p_1 = 2$ the single item is overdemanded, whereas at $p_1 = 3$ there is no demand at all for the item. □

For the situation in Case 3 also the descending bid auction proposed in Sotomayor (2002) fails to allocate the item to one of the bidders. Then the auction starts with high prices and the prices of the items in a so-called underdemanded set are decreased until there are no underdemanded sets anymore. Clearly, then at $p_1 = 3$ the single item is still underdemanded, but decreasing the price from 3 to 2 results in an overdemand for the item. In Case 3, not only a Walrasian equilibrium does not exist, but also the existing auction mechanisms fail to allocate the item. A way out of such situations that an item is overdemanded at some price and not demanded at the next price is by allotting the item to one of the bidders who demands the item at the highest price with overdemand, for instance, by having a lottery between these bidders. This bidder has to pay the highest price at which he demanded the item. Of course, allotting the item to one of these bidders implies that the item cannot be assigned to the others who demanded also the item at the same price. So, the auctioneer can only accept one of the bids but has to decline all other equal bids. In the last case of Example 1, the auctioneer might accept the bid $p_1 = 2$ of one of the bidders, while declining the equal bid of the other. This leaves the latter bidder with the dummy item which yields him a lower utility than the net surplus $V^i(1) - p_1$ that he derives from item 1 against price $p_1 = 2$. Depending on whether the item is sold to the second or first bidder, the resulting outcome can or cannot be socially efficient. Note, however, that even allocating the item to the first bidder gives an efficiency improvement compared with the situation that the item remains at the seller.

It is well-known that when there are price rigidities or fixed prices, the Walrasian equilibrium usually fails to exist. In such situations, rationing is widely used to facilitate the allocation of goods or services; see for instance, Drèze (1975), Cox (1980), van der Laan (1980), Kurz (1982), Azariadis and Stiglitz (1983) on divisible goods, and Talman and Yang (2006) on indivisible goods. Typically the allocation that results from rationing is not efficient. In a market situation with financially constrained bidders the necessity to decline bids of some bidders while accepting an equal bid of one bidder also results in a situation of rationing. After all, any bidder who leaves the auction with a net surplus lower than the net surplus that could have been obtained from an item j that was allotted to some other bidder feels himself a posterior rationed on the demand of such an item j. To explore this observation, we will adapt the Walrasian equilibrium by incorporating the concept of an *allotment scheme* $R = (R^1, \cdots, R^m)$ where, for $i \in M$, the vector $R^i \in \{0,1\}^{n+1}$ is a *rationing* vector yielding which goods bidder i can demand and for which goods offers of bidder i will be declined. That is, $R^i_j = 1$ means that bidder i is allowed to demand good j, while $R^i_j = 0$ means that bidder i is not allowed to demand good $j \in N$. Since the dummy item is always available for every bidder i, we have that $R^i_0 = 1$ for all i. Given a rationing vector R^i with $R^i_j = 0$ for item j, the vector R^i_{-j} denotes the same R^i but allows bidder i to demand item j by ignoring $R^i_j = 0$. At a feasible price vector p and an allotment scheme $R = (R^1, \cdots, R^m)$, the constrained demand set of bidder $i \in M$ is given by

$$\begin{aligned} D^i(p, R^i) \quad = \quad & \{ j \in N \mid R^i_j = 1,\ p_j \leq m^i \text{ and} \\ & V^i(j) - p_j = \max_{\{k \in N \cup \{0\} \mid p_k \leq m^i \text{ and } R^i_k = 1\}} (V^i(k) - p_k) \}. \end{aligned}$$

Now we present the solution concept of rationed equilibrium to the current model with financially constrained bidders.

Definition 2.2 *A* rationed equilibrium (**RE**) (p, π, R) *on a market with financially constrained bidders consists of an implementable pair* (p, π) *and an allotment scheme* R *such that*

(i) $\pi(i) \in D^i(p, R^i)$ *for all* $i \in M$;

(ii) $p_j = c(j)$ *for any unassigned item* $j \in N_\pi$;

(iii) *If* $R^i_j = 0$ *for some* i, *then* $\pi(h) = j$ *for some* $h \in M \setminus \{i\}$;

(iv) *If* $R^i_j = 0$, *then* $j \in D^i(p, R^i_{-j})$;

(v) *If* $\pi(h) = j$ *and there exists* $i \neq h$ *with* $R^i_j = 0$, *then* $m^h = p_j$.

Conditions (i) and (ii) correspond to Conditions (a) and (b) of the definition of the Walrasian equilibrium and are straightforward. In (iii)-(v) conditions on the allotment scheme are specified. First, Condition (iii) states that rationing on an item can only occur if the item is assigned to some bidder. So, rationing cannot occur when the item is not sold. Condition (iv) says that any rationing is binding, i.e., if a bidder is rationed on an item, then he will demand the item if the rationing on that item is dropped. Finally, Condition (v) states that when some bidder is rationed on his demand of an item j, then the bidder who receives the item has to pay all his money for the item and thus cannot afford a higher price.

A rationed equilibrium (p, π, R) is said to be *unrationed* if no bidder is constrained on his demand of any item, i.e., $R^i_j = 1$ for all $i \in M$ and $j \in N$. An unrationed equilibrium is just a Walrasian equilibrium.

3 Basic properties of over and underdemanded sets

To design an auction that can also deal with financially constrained bidders, we need to find a way around the disequilibrium situation. As discussed before, a possibility to solve the disequilibrium problem in the third case $m^1 = m^2 = 2$ of Example 1 is to sell the item to one of the two bidders at price $p_1 = 2$. To implement this in an ascending auction, the auctioneer has to decide on selling the item at the previous price $p_1 = 2$ when she observes that the item is not demanded anymore at the price $p_1 = 3$. The auction we design in the next section will have this feature as one of its rules. The auctioneer starts by announcing the seller's reservation prices of the real items and requires the bidders to respond with their demand sets. When there is an overdemanded set of items, the prices of the items in a minimal overdemanded set are increased with one and the bidders are required again to report their demand sets. This continues until there is no overdemand anymore. Then either a situation is reached in which there is an underdemanded set of items, or there is neither over nor underdemand. In the latter case the auction stops, in the first case precisely one of the items in the chosen minimal overdemanded set in the previous round is sold against its price in the previous round, after which the auction continues with the remaining bidders by announcing again the previous round prices of the remaining items.

The notion of minimal overdemanded sets is introduced in Demange, Gale and Sotomayor (1986) and generalized in Gul and Stachetti (2000), and the notion of underdemanded sets can be found in Sotomayor (2002) (see also Mishra and Veeramani (2006) and Mishra and Talman (2006)). In this section we derive several important properties of overdemanded and underdemanded sets that will be used for proving the convergence of the proposed auction. Let $|A|$ stand for the cardinality of a finite set A.

For a set of real items $S \subseteq N$, and a price vector $p \in \mathbb{R}^{n+1}_+$, define the *lower inverse demand* set of S at p by

$$D^-_S(p) = \{i \in M \mid D^i(p) \subseteq S\},$$

i.e., this is the set of bidders who demand only items in S. We also define the *upper inverse demand* of S at p by

$$D^+_S(p) = \{i \in M \mid D^i(p) \cap S \neq \emptyset\},$$

i.e., this is the set of bidders that demand at least one of the items in S. Clearly, the lower inverse demand set is a subset of the upper inverse demand set.

Definition 3.1
1. *A set of real items $S \subseteq N$ is overdemanded at price vector $p \in \mathbb{R}_+^{n+1}$ if $|D_S^-(p)| > |S|$.*
2. *A set of real items $S \subseteq N$ is underdemanded at price vector $p \in \mathbb{R}_+^{n+1}$ if*
(i) $S \subseteq \{j \in N \mid p_j > c(j)\}$; *and* (ii) $|D_S^+(p)| < |S|$.

The definition says that a set of real items $S \subseteq N$ is overdemanded at p if the number of bidders who demand only items in S is strictly greater than the number of items in S. Observe that S is a subset of real items, so any bidder i in the lower inverse demand set does not demand the dummy item and thus has a strict positive surplus $V^i(j) - p_j$ for any item j in his demand set $D^i(p)$. An overdemanded set S is said to be *minimal* if no strict subset of S is overdemanded. A set of real items $S \subseteq N$ is underdemanded at p if the price of every item in S is strictly greater than the seller's reservation price and the number of bidders who demand at least one item in S is strictly less than the number of items in S. An underdemanded set S is said to be *minimal* if no strict subset of S is an underdemanded set. We further say that an item j is *overpriced* if $D_{\{j\}}^+(p) = \emptyset$, i.e., no bidder has item j in his demand set. Observe that $S = \{j\}$ is a minimal underdemanded set when item j is overpriced. So, a minimal underdemanded set S either contains at least two (not overpriced) items, or has an overpriced item as its single element.

To give a characterization of a minimal overdemanded set, we first consider an example.

Example 2. There are three real items ($n = 3$) and five bidders ($m = 5$). Suppose that at some $p \in \mathbb{R}_+^4$, $D^1(p) = \{1, 2, 3\}$, $D^2(p) = \{1, 2\}$, $D^3(p) = \{2, 3\}$, $D^4(p) = \{1\}$ and $D^5(p) = \{3\}$. Then $S = \{1, 2, 3\} = N$ is a minimal overdemanded set with $D_S^-(p) = \{1, 2, 3, 4, 5\} = M$. Observe that indeed there is no strict subset of S that is overdemanded. Further, observe that any single item in S is in the demand set of (at least) three bidders, which number is equal to one plus the difference between the number of bidders in the lower inverse demand set of S and the number of items in S. □

The latter observation appears to be a general property, that will play an important role later in the proof that the auction terminates in a finite number of steps. The next lemma states a more general property that for every nonempty subset S of a minimal overdemanded set O at p, the number of bidders in the lower inverse demand set $D_O^-(p)$ that demand at least one item of S is at least equal to the number of items in S plus the difference between $|D_O^-(p)|$ and $|O|$ and thus is at least one more than the number of items in S.

Lemma 3.2 *Let O be a minimal overdemanded set of items at a price vector p. Then, for every nonempty subset S of O, we have*

$$|\{i \in D_O^-(p) \mid D^i(p) \cap S \neq \emptyset\}| \geq |S| + |D_O^-(p)| - |O|.$$

Proof. Since O is overdemanded at p, the constant $d = |D_O^-(p)| - |O|$ must be a positive integer. By definition the lemma holds (with equality) for $S = O$. For any nonempty strict subset S of O, define $D_S = \{i \in D_O^-(p) \mid D^i(p) \cap S \neq \emptyset\}$. Then we have

$$D_O^-(p) \setminus D_S = \{i \in D_O^-(p) \mid D^i(p) \subseteq O \setminus S\}.$$

Suppose to the contrary that $|D_S| < |S| + d$. Since $0 < |S| \leq |O| - 1$, we have that

$$\begin{aligned} |D_O^-(p) \setminus D_S| &= |D_O^-(p)| - |D_S| > |D_O^-(p)| - (|S| + d) = \\ &= |D_O^-(p)| - |S| - (|D_O^-(p)| - |O|) = |O| - |S| = |O \setminus S|. \end{aligned}$$

This means that the set $O \setminus S$ is overdemanded, contradicting the fact that O is a minimal overdemanded set. Hence, $|D_S| \geq |S| + d = |S| + |D_O^-(p)| - |O|$. □

The next corollary follows immediately.

Corollary 3.3 *For every item in a minimal overdemanded set O at p, there are at least two bidders in $D_O^-(p)$ (actually the number is $|D_O^-(p)| - |O| + 1 \geq 2$) demanding that item.*

The properties in the next lemma and corollary are not needed for proving the convergency of the auction. Nevertheless, we state these properties because they are interesting in themselves. Considering Example 2 once more, let us assign one of the items in the demand set $D^2(p) = \{1, 2\}$, say item 1, to bidder 2. Then we claim that given this assignment, there exists an assignment of all the remaining items in the set O to the remaining bidders in $D_O^-(p)$ such that (i) every bidder $D_O^-(p) \setminus \{2\}$ is assigned at most one item from $O \setminus \{1\}$ and (ii) every item j in $O \setminus \{1\}$ is assigned to precisely one bidder h in $D_O^-(p) \setminus \{2\}$ satisfying that $j \in D^h(p)$. For example, we can assign item 2 to bidder 3 and item 3 to bidder 5. The claim follows from the following lemma that shows a general property of minimal overdemanded sets. The proof makes use of a well-known combinatorial theorem of Hall (1935).

Lemma 3.4 *Let O be a minimal overdemanded set at a price vector p and let T be any subset of $D_O^-(p)$ such that $|T| = |O|$. Then there exists an assignment of items in O to the bidders in T such that every bidder in T is assigned at most one item, and every item in O is assigned to precisely one bidder in T having that item in his demand set.*

Proof. Since $|O|$ is overdemanded, we have $|O| < |D^-_O(p)|$. Let T be any subset of $D^-_O(p)$ with $|T| = |O|$. We first show that for every subset S of T, the number of items demanded by at least one bidder in S is at least as great as the number of bidders in S. Let $D(S) = \cup_{i \in S} D^i(p)$ be the set of items demanded by at least one bidder in S. Suppose to the contrary that $|D(S)| < |S|$. Note that for every $i \in S$, $D^i(p) \subseteq D(S) \subseteq O$ and $|D(S)| < |S| \leq |T| = |O|$. This implies that $D(S)$ is both a strict subset of the minimal overdemanded set O and an overdemanded set, contradicting the fact that O is a minimal overdemanded set. So we have $|D(S)| \geq |S|$ for all $S \subseteq T$ and thus in particular $|D(T)| = |T|$ because $D^h(p) \subseteq O = T$ for every $h \in T \subset D^-_O(p)$. Now it follows immediately from Hall's theorem that there exists a bijective mapping π between O and T such that $\ell \in D^{\pi(\ell)}(p)$ for every $\ell \in O$. □

The lemma implies the following corollary, which shows the claim above.

Corollary 3.5 *Let O be a minimal overdemanded set at a price vector p, let k be an item in O and let h be a bidder in $D^-_O(p)$ having k in his demand set. Then there exists an assignment of items in O to the bidders in $D^-_O(p)$ such that* (i) *item k is assigned to bidder h,* (ii) *every bidder is assigned at most one item, and* (iii) *every item is assigned to precisely one bidder having that item in his demand set.*

Proof. Since $|D^-_O(p)| > |O|$, there exists $T \subset D^-_O(p)$ such that $|T| = |O|$ and $h \notin T$. According to Lemma 3.4 there exists a bijective mapping π between O and T such that $\ell \in D^{\pi(\ell)}(p)$ for every $\ell \in O$. Let h' be the bidder in T who is assigned item k at π, i.e., $\pi(k) = h'$. Now, let ρ be the assignment of items in O to the bidders in $D^-_O(p)$ defined by $\rho(k) = h$ and $\rho(j) = \pi(j)$ for any $j \in O \setminus \{k\}$, i.e., the item k that was assigned to h' in π is now assigned to bidder h. Clearly, the assignment ρ satisfies the conditions (i), (ii) and (iii). □

The next result shows that the number of bidders in the upper inverse demand set of a minimal underdemanded set is precisely one less than the number of items in the set and that any bidder in the upper inverse demand set demands at least two items from the minimal underdemanded set.

Lemma 3.6 *Let U be a minimal underdemanded set of items at a price vector p. Then $|D^+_U(p)| = |U| - 1$ and the demand set $D^i(p)$ of every bidder $i \in D^+_U(p)$ contains at least two elements of U.*

Proof. If $|U| = 1$, then U consists of an overpriced item and $|D^+_U(p)| = 0$. So, both statements are true.

For $|U| \geq 2$, denote $T = D^+_U(p)$. To prove the first part, suppose $|T| \leq |U| - 2$. Then take any element k of U and denote $T' = D^+_{U\setminus\{k\}}(p)$. Clearly, $T' \subseteq T$ and thus $|T'| \leq |T|$. Hence

$$|T'| \leq |T| \leq |U| - 2 = |U \setminus \{k\}| - 1$$

and thus $U \setminus \{k\}$ is underdemanded, contradicting the assumption that U is a minimal underdemanded set.

To prove the second part, suppose there is a bidder i having only one element of U in his demand set. Let k be this element. Then T' does not contain bidder $i \in T$. Hence $|T'| \leq |T| - 1$ and thus

$$|T'| \leq |T| - 1 = |U| - 2 = |U \setminus \{k\}| - 1,$$

showing that $U\setminus\{k\}$ is underdemanded. Again this contradicts the fact that U is a minimal underdemanded set. □

The following lemma is given by Mishra and Talman (2006, Theorem 1) for the case without financial constraints. In fact, the lemma is only concerned with the demand sets of the bidders, no matter whether these demands are under financial constraints or not. Here we provide a much simpler proof of the lemma, by using Hall's theorem twice.

Lemma 3.7 *There is a Walrasian equilibrium at $p \in \mathbb{R}^{n+1}_+$ if and only if at p no set of items is overdemanded and no set of items is underdemanded.*

Proof. First, let (p, π) be a Walrasian equilibrium (p, π). Clearly, at p no set of items is overdemanded and no set of items is underdemanded.

To prove the other direction, let $M^1 = \{i \in M \mid 0 \notin D^i(p)\}$ and $N^1 = \{j \in N \mid p_j > c(j)\}$. First, consider any $T \subseteq M^1$ and let $D^T = \cup_{i\in T} D^i(p)$. Because D^T is not overdemanded, $|D^T| \geq |T|$. By Hall's Theorem, there exists a one-to-one mapping $\tau\colon M^1 \to N^1$ such that $\tau(i) \in D^i(p)$ for all $i \in M^1$. We can extend τ to a mapping from M to $N \cup \{0\}$ by setting $\tau(i) = 0$ for all $i \notin M^1$. Next, consider any $S \subseteq N^1$. Because S is not underdemanded, $|D^-_S(p)| \geq |S|$. Again by Hall's Theorem, there exists a one-to-one mapping $\rho : N^1 \to M$ such that $j \in D^{\rho(j)}(p)$ for all $j \in N^1$.

With respect to τ and ρ, denote $K = \{i \mid \tau(i) \in N^1\}$, $L = \{\tau(i) \mid i \in K\}$ and $Q = \{\rho(j) \mid j \in N^1 \setminus L\}$ and define the mapping $\pi\colon M \to N \cup \{0\}$ by

$$\pi(i) = \{ \begin{array}{ll} \tau(i), & \text{for } i \in M \setminus Q, \\ \rho^{-1}(i), & \text{for } i \in Q. \end{array}$$

Clearly, $\pi(i) \in D^i(p)$ for all $i \in M$, and no real item is assigned by π to two different bidders, and for every item $j \in N^1$, there is a bidder i who demands the item at p and is assigned the item. This shows that (p, π) is a Walrasian equilibrium. □

4 The auction design

In this section we design an ascending auction for a market with financially constrained bidders that always results in an allocation of the items within a finite number of steps. In the next section we discuss the properties of the resulting allocation and price system and show that the outcome yields a rationed equilibrium. Since during the auction process the set of bidders and the set of items are shrinking, the notions of price vector, demand set and (minimal) overdemanded and underdemanded sets all have to be adapted accordingly. Let $C = (c(0), c(1), \cdots, c(n))^\top$ be the vector of the seller's reservation prices. It should be noted that the demand set of bidder h at stage t of the auction is given by

$$D^h(p^t) = \{j \in N^t \cup \{0\} \mid p^t_j \leq m^i,\ V^i(j) - p^t_j = \max_{\{k \in N^t \cup \{0\} \mid p^t_k \leq m^i\}} (V^i(k) - p^t_k)\},$$

where $N^t \subseteq N$ is the set of available real items and p^t is the vector of prices of the items in $N^t \cup \{0\}$ at stage t.

The Ascending Auction

Step 1 (Initialisation): Set $t := 1$, $p^t := C$, $N^t := N$ and $M^t := M$. Go to Step 2.

Step 2: Every bidder $i \in M^t$ reports his demand set $D^i(p^t) \subseteq N^t \cup \{0\}$, being the set of the most preferred items demanded by bidder i from the set $N^t \cup \{0\}$ at prices p^t. If there exists an underdemanded set at p^t, go to Step 3. Otherwise, if there is an overdemanded set at p^t, the auctioneer chooses a minimal overdemanded set O^t of items. Then set $p^{t+1}_j := p^t_j + 1$ for every $j \in O^t$, $p^{t+1}_j := p^t_j$ for every $j \in (N^t \setminus O^t) \cup \{0\}$, $M^{t+1} := M^t$ and $N^{t+1} := N^t$. Set $t := t + 1$ and return to Step 2. When there are no overdemanded set of items and no underdemanded set of items at p^t, the auction stops.

Step 3: Let U^t be a minimal underdemanded set. Then take some item $k \in U^t \cap O^{t-1}$ and bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ such that $k \in D^h(p^{t-1})$, but $k \notin D^h(p^t)$ and assign item k to bidder h against price p^{t-1}_k. Set $M^{t+1} := M^t \setminus \{h\}$ and $N^{t+1} := N^t \setminus \{k\}$. If $N^{t+1} = \emptyset$, the auction stops, otherwise let $p^{t+1}_j := p^{t-1}_j$ for all $j \in N^{t+1} \cup \{0\}$. Set $t := t + 1$ and return to Step 2.

For convenience, the parameter t in the auction will be referred to as *stage* t of the auction. Note that at any stage t the price of the dummy item is equal to $p^t_0 = c(0) = 0$. Before considering the feasibility and convergence of the auction, we first discuss the steps in more detail and then provide an example to illustrate how the auction actually operates.

In Step 1, the auctioneer announces a set of items for sale and sets the starting prices equal to the reservation prices, and a group of bidders come to bid.

In Step 2, each bidder is asked to report his demand set for the available items at the current prices. Based on the reported demands from the bidders, the auctioneer first checks if there is any underdemanded set of items. If so, then Step 3 will be performed. Otherwise, the auctioneer checks whether there is any overdemanded set of items. If not, the auction stops. According to Lemma 3.7, in this case a Walrasian equilibrium has been reached for the remaining set of items and bidders. In case there is overdemand, the auctioneer chooses a minimal overdemanded set of items and goes to the next stage. In this stage the price of every item in the chosen minimal overdemanded set is increased by one, the price of any other item remains constant and Step 2 will be performed again.

In Step 3, the auctioneer first chooses a minimal underdemanded set. Then she selects precisely one item, say item k, that belonged to the minimal overdemanded set that was chosen in Step 2 at the previous stage $t-1$ and that also belongs to the minimal underdemanded set at the current stage t. This item k is assigned to a bidder h satisfying (i) who was in the lower inverse demand set of the minimal overdemanded set chosen in stage $t-1$, (ii) who demanded the item k at the previous stage $t-1$, and (iii) who does not demand item k anymore at the current stage t. This bidder h pays the price p_k^{t-1} of item k at the previous stage and leaves the auction with the item k. The auction moves to the next stage $t+1$ with the remaining items and bidders. When no real items are left, the auction stops. Otherwise the prices of all the remaining items are set equal to the prices in stage $t-1$ and Step 2 will be performed again.

It should be noticed that in Step 3 it can never occur that there are no remaining bidders. Clearly, this is true when the number of bidders m is larger than the number of items n, because in Step 3 always precisely one bidder leaves with one item. When $m \leq n$, it might happen that at certain stage the auction returns from Step 3 to Step 2 with only one bidder. Obviously then overdemand cannot occur in Step 2. In the sequel we prove that underdemand can never occur in Step 2 when the auction returned from Step 3 in the previous stage. So, when after Step 3 the auction returns to Step 2 with precisely one bidder, then neither underdemand nor overdemand can occur and the auction terminates in Step 2.

Example 4. Consider a market with five bidders (1, 2, 3, 4, 5) and four real items (1, 2, 3, 4). Bidders know their own values and budgets privately but the seller does not have this information. The initial endowment vector of money is given by $m = (m^1, m^2, m^3, m^4, m^5) = (3, 4, 3, 5, 4)$ and bidders' values are given in Table 1. The seller's reservation price vector is given by $C = (c(0), c(1), c(2), c(3), c(4)) = (0, 2, 2, 2, 2)$. She will not reveal the reservation price of any real item until the auction starts.

Table 1: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	8	5	7
Bidder 2	0	7	6	8	3
Bidder 3	0	5	5	9	7
Bidder 4	0	9	4	6	2
Bidder 5	0	6	5	4	10

This market has a unique socially efficient allocation $\pi^* = (\pi^*(1), \pi^*(2), \pi^*(3), \pi^*(4), \pi^*(5)) = (2, 0, 3, 1, 4)$, which gives a total value of $\sum_{i \in M} V^i(\pi^*(i)) = 36$. Without financial constraints, the ascending DGS auction would end at stage 7 with the minimal equilibrium price vector $p^* = (p_0, p_1, p_2, p_3, p_4) = (0, 7, 6, 8, 6)$ and the socially efficient allocation π^*. The seller's revenue generated by the auction is 27. However, in the current situation with financial constraints, the bidders cannot afford to buy items at these minimal equilibrium prices and so a Walrasian equilibrium does not exist. In this situation the standard ascending auction generates at stage 6 the price vector $p = (0, 3, 3, 4, 4)$ (see stage 6 in Table 2 below). At this price vector there is overdemand for the items 1 and 2 (both items are demanded by three bidders) and the prices of the items 1 and 2 are increased in the next stage. However, at the new price vector $p = (0, 4, 4, 4, 4)$ there is no demand anymore for item 2, i.e., item 2 is overpriced, and the auction breaks down without reaching an equilibrium (see stage 7 in Table 2 below). Of course, in this final stage 7 it is still possible to allocate item 1 to the unique bidder 4 having 1 in his demand set, item 3 to the unique bidder 2 and item 4 to the unique bidder 5. However, item 2 is not allocated and the remaining bidders 1 and 3 don't get any real item. The resulting allocation gives a total value of $V^2(3) + V^4(1) + V^5(4) + c(2) = 29$ and is not socially efficient. The seller's revenue from the auction is only 12 and her total revenue is $12 + c(2) = 14$.

The new ascending auction described above yields a rationed equilibrium with both a higher total value and higher revenue for the seller. The auction starts at stage $t = 1$ with for the real items the price vector $p^1 = (0, 2, 2, 2, 2)$. Then bidders report their demand sets: $D^1(p^1) = \{2\}$, $D^2(p^1) = \{3\}$, $D^3(p^1) = \{3\}$, $D^4(p^1) = \{1\}$ and $D^5(p^1) = \{4\}$. The set $S = \{3\}$ is a minimal overdemanded set and the auctioneer adjusts p^1 to $p^2 = (2, 2, 3, 2)$ at stage $t = 2$. The price vectors, demand sets and other relevant data generated by the auction are given in Table 2. Since $p_0^t = 0$ for all t, these prices are deleted from the vectors p^t in the second column.

At stage 7, item 2 is overpriced and bidders 1 and 3 cannot afford any real item because of their financial constraints. The auctioneer assigns item 2 randomly to one of these bidders, say to bidder 1, against price $p_2^6 = 3$. So bidder 1 leaves the auction with item 2 and at the next stage 8 we have that $M^8 = \{2, 3, 4, 5\}$ and $N^8 = \{1, 3, 4\}$. The

Table 2: The data generated by the auction in Example 4.

Stage	p^t	N^t	M^t	$D^1(p^t)$	$D^2(p^t)$	$D^3(p^t)$	$D^4(p^t)$	$D^5(p^t)$	O^t
1	$(2,2,2,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$	$\{3\}$
2	$(2,2,3,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,3\}$	$\{3\}$	$\{1\}$	$\{4\}$	$\{1,3\}$
3	$(3,2,4,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{4\}$	$\{1\}$	$\{4\}$	$\{4\}$
4	$(3,2,4,3)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{4\}$	$\{1\}$	$\{4\}$	$\{4\}$
5	$(3,2,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{2\}$	$\{1\}$	$\{4\}$	$\{2\}$
6	$(3,3,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{1\}$	$\{4\}$	$\{1,2\}$
7	$(4,4,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{0\}$	$\{3\}$	$\{0\}$	$\{1\}$	$\{4\}$	
8	$(3,4,4)$	$\{1,3,4\}$	$\{2,3,4,5\}$		$\{1,3\}$	$\{1\}$	$\{1\}$	$\{4\}$	$\{1\}$
9	$(4,4,4)$	$\{1,3,4\}$	$\{2,3,4,5\}$		$\{3\}$	$\{0\}$	$\{1\}$	$\{4\}$	

auctioneer adjusts p^7 to $p^8 = (p_1, p_3, p_4) = (3, 4, 4)$, being for these three real items the same prices as at stage 6. Then at stage 8 item 1 is overdemanded and its price is increased to $p_1 = 4$ at stage 9. At this stage there is neither overdemanded set nor underdemanded set. The auction stops in Step 2 (i.e., stage 9) with, according to Lemma 3.7, a Walrasian equilibrium for the sets of remaining items and bidders. Indeed, the auctioneer can assign the dummy item 0 to bidder 3 (who pays nothing), and the items 1, 3 and 4 to the bidders 4, 2 and 5 respectively, against the prices $p^9 = (p_1, p_3, p_4) = (4, 4, 4)$. The final price system $p^* = (p_0, p_1, p_2, p_3, p_4) = (0, 4, 3, 4, 4)$ and allocation $\pi^* = (\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)) = (2, 3, 0, 1, 4)$ yields a rationed equilibrium with allotment scheme $R^* = (R^1, R^2, R^3, R^4, R^5)$, where $R_3^{*3} = 0$ and $R_j^{*i} = 1$ for all $(i, j) \neq (3, 3)$. The allocation in this rationed equilibrium yields a total value of $\sum_{i \in M} V^i(\pi(i)) = 35$, which is slightly less than the value 36 of the Walrasian equilibrium allocation. Even when in stage 7 item 2 should have been assigned to bidder 3, the auction would still realise a total value of 33. In both cases the seller's revenue from the auction is 15, which is also equal to her total revenue, because all items are sold. □

Now we will prove that the auction is well-designed. That is to show, all steps are feasible and the auction stops in finitely many steps. First, observe that in each Step 3 an item is assigned to one bidder and both the set of bidders and the set of items decrease by one. When $m \leq n$, at each stage t we have that $|M^t| \leq |N^t|$. We will show that in this case the auction always stops in Step 2. When $m > n$, then at each stage t we have that $|M^t| > |N^t|$. In this case the auction stops either in Step 3 when $N^{t+1} = \emptyset$ or in Step 2. When the auction stops in Step 3 all items are assigned sequentially in n Steps 3. When the auction stops at some stage t in Step 2, no overdemanded or underdemanded sets are left at the price p^t, and according to Lemma 3.7, there exists a feasible allocation $\pi^t \colon M^t \to N^t$ such that (p^t, π^t) is a Walrasian equilibrium for the current sets M^t and N^t at stage t.

Clearly, Steps 1 and 2 are feasible. So to show that the auction is well-designed, we only need to consider Step 3. Observe that we start with all prices equal to the seller's reservation prices. At this starting price system there is no underdemand, because by Definition 3.1.2 an item can only be underdemanded when its price is above its seller's reservation price. So, at the starting price p^1 at stage $t = 1$, either the auction stops or there is overdemand. In the latter case, a sequence of Steps 2 is performed with at each stage an increase of the prices of all items in a minimal overdemanded set, until items become underdemanded, say at stage t, and the auction goes to Step 3. So, when at some stage t the auction goes to Step 3 for the first time, then in stage $t-1$ the prices in some minimal overdemanded set, say O^{t-1}, were increased. We will prove that this holds at any stage t in which the auction goes to Step 3, i.e., when there is underdemand at some stage t, then there was overdemand at stage $t-1$ and thus, when the auction reaches Step 3 at some stage t, then at stage $t-1$ the prices of the items in some minimal overdemanded set O^{t-1} were increased. In Step 3 an item k in the intersection of some minimal underdemanded set U^t and the set O^{t-1} is selected and assigned to a bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ satisfying $k \in D^h(p^{t-1}) \setminus D^h(p^t)$. The next two lemmas show that there indeed exist such an item k and bidder h. It should be noticed that in all proofs the sets $D^-_S(p^\tau)$ and $D^+_S(p^\tau)$ are defined with respect to the current set of bidders M^τ, for any set $S \subset N^\tau$ and for any $\tau = t-1,\ t$.

Lemma 4.1 *Let U be a minimal underdemanded set at prices p^t in some stage t and let O be the chosen minimal overdemanded set at the previous stage $t-1$. Then $U \cap O \neq \emptyset$.*

Proof. Suppose to the contrary that $U \cap O = \emptyset$. Since U is underdemanded at stage t, we have that $p^t_j > c(j)$ for any $j \in U$. Further, since $U \cap O = \emptyset$, we have for any $j \in U$ that $j \notin O$. Hence $p^t_j = p^{t-1}_j$ and thus also $p^{t-1}_j > c(j)$ for all $j \in U$. Since there is no underdemand at stage $t-1$, it follows that $|D^+_U(p^{t-1})| \geq |U|$. Moreover, any bidder that demands some item $j \in U$ at p^{t-1}, also demands this item at p^t, because only prices of the items in O are increased. Hence $|D^+_U(p^t)| \geq |D^+_U(p^{t-1})| \geq |U|$ and thus U is not underdemanded at p^t, yielding a contradiction. Hence $U \cap O \neq \emptyset$. □

Lemma 4.2 *Let U be a minimal underdemanded set at prices p^t in some stage t and let O be the chosen minimal overdemanded set at the previous stage $t-1$. Then there exist item k and bidder h satisfying the requirements of Step 3.*

Proof. Since O is overdemanded at p^{t-1}, we have $|D^-_O(p^{t-1})| > |O|$. Now, consider the set $S = U \cap O$. By Lemma 4.1 this set is not empty. When $U = O$ and thus $S = O$, then by Lemma 3.6 there are $|U| - 1 = |O| - 1$ bidders demanding at least one item from U at

p^t, because U is underdemanded at p^t. So, in this case there are at least two bidders in $D_O^-(p^{t-1})$ not demanding any item from $U = O$ anymore at price p^t. Select h from this set of bidders and select k from the set $D^h(p^{t-1})$ (recall that this set is never empty and does not contain any dummy item). Since $D^h(p^{t-1}) \subseteq O$ and for each bidder $h \in D_O^-(p^{t-1})$, this item k and this bidder h satisfy the requirements.

Next, consider the case that S is a strict subset of O. Denote $H = \{i \in D_O^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S \neq \emptyset\}$. From Lemma 3.2 we have that

$$|H| \geq |S| + |D_O^-(p^{t-1})| - |O| \geq |S| + 1,$$

i.e., the number of bidders in $D_O^-(p^{t-1})$ that demand an item of S at p^{t-1} is at least one more than the number of items in S. Next, consider the set $T = U \setminus O$. Since there is no underdemand at p^{t-1} we have that

$$|D_T^+(p^{t-1})| \geq |T|.$$

Since $p_j^t = p_j^{t-1}$ for all $j \in T = U \setminus O$, any bidder that demands an item from T at p^{t-1}, is still demanding this item at p^t, so $D_T^+(p^{t-1}) \subseteq D_T^+(p^t)$. On the other hand, U is underdemanded at p^t, so

$$|D_U^+(p^t)| < |U|.$$

Further, observe that $H \cap D_T^+(p^{t-1}) = \emptyset$, since $H \subseteq D_O^-(p^{t-1})$ and the members of $D_O^-(p^{t-1})$ demand only items in O, whereas the members of $D_T^+(p^{t-1})$ demand at least one item from $T = U \setminus O$ at p^{t-1}. Therefore, the number of bidders in H that still demand items in S at p^t can be at most $|S| - 1$. Suppose not, i.e., the number is at least $|S|$. Then the number of bidders in $D_U^+(p^t)$ (demanding at least one item of U at p^t) is at least equal to $|S|$ plus the number of bidders in $D_T^+(p^{t-1})$, i.e.

$$|D_U^+(p^t)| \geq |S| + |T| = |U \cap O| + |U \setminus O| = |U|,$$

contradicting the fact that U is underdemanded. Hence there are at least two bidders in H that are no longer demanding items in $U \cap O$ at p^t. Select h as one of these bidders and k as one of the elements in the non-empty set $D^h(p^{t-1}) \cap S$. Then item k and bidder h satisfy the requirements. $\square$

In the special case that $U = \{k\}$ with $k \in O$, i.e., the single item k in U is overpriced, we have that no bidder is demanding k at p^t. Hence, any bidder h in $D_O^-(p^{t-1})$ having k in his demand set at p^{t-1} can be selected. Note that according to Corollary 3.3 there are at least two of such bidders.

In the next lemma we prove that any time when at some stage $t + 1$ the auction enters Step 2 after at stage t an item k has been assigned to some bidder h in Step 3, there will be no underdemand of items. So, when the auction arrives in Step 2 after Step 3, either there is neither underdemand nor overdemand and the auction stops, or there is overdemand and the prices of items in some minimal overdemanded set are increased. This guarantees that any time when the auction goes to Step 3, prices in some minimal overdemanded set were increased in the previous stage. Recall that when at stage $t+1$ Step 2 is reached from Step 3, the price vector p^{t+1} is equal to the price vector p^{t-1}, except that item k has been deleted.

Lemma 4.3 *Let U be a minimal underdemanded set at stage t that appears after at stage $t-1$ the prices of the items in a minimal overdemanded set O were increased, and let, in Step 3, $k \in U \cap O$ be the item assigned to some bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O\}$ such that $k \in D^h(p^{t-1})$, but $k \notin D^h(p^t)$. When the auction proceeds to stage $t+1$ and returns to Step 2, then there will be no underdemanded set of items.*

Proof. First, observe that, by definition of the auction, $M^{t+1} = M^{t-1} \setminus \{h\}$, $N^{t+1} = N^{t-1} \setminus \{k\}$ and $N^{t+1} \neq \emptyset$ (otherwise the auction ends in Step 3). Further, $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$. Denote $\tilde{O} = O \setminus \{k\}$. For $S \subseteq N^{t+1}$ we consider two cases, namely $S \subseteq \widetilde{O}$ and $S \setminus \widetilde{O} \neq \emptyset$. In the first case we have by Lemma 3.2 that at least $|S|+1$ members of the set $D_O^-(p^{t-1}) = \{i \in M^{t-1} \mid D^i(p^{t-1}) \subseteq O\}$ demanded at least one item of S in stage $t-1$. Since $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$, for any bidder i in M^{t+1} it holds that

$$D^i(p^{t+1}) = D^i(p^{t-1}) \setminus \{k\}$$

and thus any bidder $i \in M^{t+1} \cap D_O^-(p^{t-1}) = D_O^-(p^{t-1}) \setminus \{h\}$ that demanded an item of S at stage $t-1$ is still demanding an item of S at stage $t+1$. So, when h demanded an item of S at stage $t-1$, the number of bidders of M^{t+1} demanding an item of S at stage $t+1$ is at least $|S|$, otherwise the number is at least $|S|+1$. Hence S is not underdemanded.

For the second case $S \setminus \widetilde{O} \neq \emptyset$ we consider the partition of S given by $S^1 = S \cap \widetilde{O}$ and $S^2 = S \setminus \widetilde{O}$. Denote

$$K^1 = \{i \in D_O^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S^1 \neq \emptyset\}$$

and

$$K^2 = \{i \in M^{t-1} \mid D^i(p^{t-1}) \cap S^2 \neq \emptyset\}.$$

Since $D^i(p^{t-1}) \subseteq O$ for all $i \in D_O^-(p^{t-1})$ and $S^2 \subseteq N^{t-1} \setminus O$, it follows that $K^1 \cap K^2 = \emptyset$. Since O is a minimal overdemanded set at stage $t-1$ and there is no underdemand at stage $t-1$, we have that S^1 is neither overdemanded nor underdemanded at p^{t-1}, because it is a strict subset of O. By Lemma 3.2 we have that at least $|S^1|+1$ members of $D_O^-(p^{t-1})$ demanded at least one item of S^1 in stage $t-1$ and similarly as above it follows that at least $|S^1|$ members of $D_O^-(p^{t-1}) \setminus \{h\}$ are still demanding an item of S^1 at stage $t+1$. Furthermore, none of these bidders belong to K^2, because $D_O^-(p^{t-1}) \cap K^2 = \emptyset$. Further $|K^2| \geq |S^2|$, because there is no underdemand at stage $t-1$. Clearly, any member of K^2 is still demanding an item of S^2 at stage $t+1$, because all prices of the remaining items in N^{t+1} are equal to the prices at stage $t-1$. Therefore the number of bidders that demand at least one item of $S = S^1 \cup S^2$ is at least equal to

$$|S^1| + |K^2| \geq |S^1| + |S^2| = |S|$$

and thus S is not underdemanded at stage $t+1$. □

The final lemma states that when in Step 3 an item has been assigned, the new set of bidders M^{t+1} cannot become empty. This is obvious when the number of bidders is bigger than the number of items. However, it also holds when the set of bidders is at most equal to the number of items. The reason is that when the auction returns from Step 3 to Step 2 with precisely one bidder, the auction immediately ends in Step 2.

Lemma 4.4 *When at some stage t an item k is assigned to some bidder $h \in M^t$ at Step 3 of the auction, then $|M^{t+1}| \geq 1$.*

Proof. Each time when an item is assigned in Step 3, the number of items and the number of bidders decreases with one. Suppose that at some stage t Step 3 occurs for the kth time. As long as $k < |M| - 1$ times we have that $M^{t+1} = |M| - k > 1$. Now, suppose that $k = |M| - 1$ at some stage t. Then $M^{t+1} = 1$ and the auction returns to Step 2. According to Lemma 4.3, there is no underdemand in Step 2. However, because only one bidder is left, also overdemand cannot occur in Step 2. Hence, according to Lemma 3.7, a Walrasian equilibrium has been reached for the remaining set of items and bidders and the auction terminates in Step 2. □

We conclude this section by showing that the auction terminates in a finite number of stages.

Theorem 4.5 *All Steps of the ascending auction are feasible. Moreover the auction terminates with a feasible allocation in a finite number of stages.*

Proof. The auction starts with all prices equal to the seller's reservation prices. When there is no overdemand and no underdemand, the auction stops in Step 2. Otherwise, there is overdemand, because $p_j = c(j)$ for all j and thus, by definition, there cannot be underdemand. Now the prices of all items in a minimal overdemanded set are increased and Step 2 is repeated until the auction stops or underdemand arises for the first time. Since the value of any item to any bidder i is finite and any initial endowment m^i is also finite, this occurs within a finite number of stages. Then the auction goes to Step 3 and assigns an item k to some bidder h. By Lemmas 4.1 and 4.2 this step is feasible. After that the auction stops in Step 3 because all items are assigned or, according to Lemma 4.4 returns to Step 2 with at least one remaining bidder. According to Lemma 4.3 there is no underdemand when the auction returns to Step 2 after Step 3. Hence, either there is no overdemand and no underdemand and the auction stops, or there is overdemand again. Then in a finite number of stages, again one item is assigned in Step 3, or the auction stops at Step 2. When the auction stops in Step 3, all items are assigned to different bidders and the auction ends up with an allocation. When the auction stops in Step 2 at some stage t, according to Lemma 3.7 there is a Walrasian equilibrium allocation with respect to the remaining items in N^t and the remaining bidders in M^t and the auction ends up with an allocation. Since the number of items is finite, the auction terminates with a feasible allocation in a finite number of stages. □

5 Fundamental properties of the auction

According to Theorem 4.5 the auction finds a feasible allocation in a finite number of steps. It remains to show that the auction results in a rationed equilibrium. Let π^* be the allocation resulting from the auction, i.e., $\pi^*(i) = k$ for some $k \in N$ when bidder i was assigned an item in either Step 2 or Step 3, and $\pi^*(i) = 0$ otherwise; and let p^* be the resulting price vector, i.e., when item k is assigned, then p^*_k is the price at which item k is assigned to some bidder h, otherwise p^*_k is the price of the item when the auction stops in Step 2. Since the auction starts with the reservation price vector C, we have that $p^t_k > c(k)$ when at stage t item k is assigned in Step 3, $p^t_k \geq c(k)$ for all items $k \in N^t$ when at stage t the auction stops in Step 2, and $p^t_0 = c(0) = 0$ for all t. Hence $p^*_k = p^{t-1}_k = p^t_k - 1 \geq c(k)$ when item k is assigned at t in Step 3, $p^*_k = p^t_k \geq c(k)$ for any item k that is assigned at the final stage t in Step 2 and $p^*_0 = c(0)$ and thus p^* is feasible. Further, when a bidder gets assigned an item in either Step 2 or Step 3, then the item is in his demand set at the price the bidder has to pay and thus every bidder i can afford to buy the item $\pi^*(i)$ assigned to him. Hence (p^*, π^*) is implementable. We further define the allotment scheme R^* as follows. For $i \in M$, define R^{i*} by

$$R^{i*}_k = \begin{cases} 0 & \text{if } k \in \{j \in N \setminus \pi^*(i) \mid p^*_j \leq m^i \text{ and } V^i(j) - p^*_j > V^i(\pi^*(i)) - p^*_{\pi^*(i)}\}, \\ 1 & \text{otherwise.} \end{cases} \quad (5.1)$$

Theorem 5.1 *The implementable pair (p^*, π^*) and the allotment scheme R^* yield a rationed equilibrium (p^*, π^*, R^*).*

Proof. We have shown above that (p^*, π^*) is an implementable pair. So, it remains to prove that the conditions (i)-(v) of Definition 2.2 hold. To prove (i), first consider a bidder i that got assigned an item k in Step 3 at some stage t against price p_k^{t-1}. Then according to Step 3,

$$k \in D^i(p^{t-1}) = \{j \in N^{t-1} \mid p_j \leq m^i,\ V^i(j) - p_j = \max_{\{\ell \in N^{t-1} \cup \{0\} \mid p_\ell \leq m^i\}} (V^i(\ell) - p_\ell)\}$$

After item k has been assigned to bidder i at stage t, the auction continues with Step 2 at stage $t+1$ with the remaining set of items $N^{t+1} = N^{t-1} \setminus \{k\}$. Since at any stage $\tau \geq t+1$, $p_j^\tau \geq p_j^{t-1}$ for all $j \in N^{t+1}$, it follows that

$$V^i(k) - p_k^* \geq V^i(j) - p_j^*, \text{ for all } j \in N^{t+1} \text{ with } p_j^* \leq m^i.$$

Further, observe that any $j \in N \setminus N^{t-1}$ has been assigned in some stage $\tau \leq t-1$, before in stage t the item k is assigned to bidder i. According to (5.1) we have that $R_j^{*i} = 0$ for all $j \in N \setminus N^{t-1}$ satisfying $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence $k \in D^i(p^*, R^{*i})$. Second we consider a bidder i who was assigned item k in Step 2 at the final stage t. Such a bidder i has item k in his demand set $D^i(p^t)$ with respect to the items in N^t. Again, for any $j \in N \setminus N^t$ that was assigned before in some stage $\tau \leq t-1$, we have that $R_j^{*i} = 0$ when $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence, also in this case we have that $k \in D^i(p^*, R^{*i})$.

To prove (ii), observe that when an item k is not assigned to a bidder i, then k belongs to the set N^t when the auction stops in Step 2 in the final stage t. According to Lemma 3.7 then the auction ends with a Walrasian equilibrium allocation with respect to the remaining items in N^t and the remaining bidders in M^t. By definition of the Walrasan equilibrium we then have that $p_k^* = p_k^t = c(k)$ for any unassigned item k.

Since there is a Walrasian equilibrium for the remaining items N^t and bidders M^t when at the final stage t the auction stops in Step 2, according to (5.1), rationing only occurs for items that has been assigned in some Step 3 before the final stage t. This proves that condition (iii) holds. Further, condition (iv) also follows immediately from (5.1).

To prove (v), again observe that when for some item j we have that $\pi(h) = j$ and $R_j^{i*} = 0$ for some bidder $i \neq h$, then item j has been allocated at Step 3 before the end of the auction. Let item j be allocated at some stage t. Then item j was in a minimal overdemanded set O at p^{t-1} and for bidder h to which j is assigned it holds that (i) $h \in \{h' \in M^t \mid D^{h'}(p^t) \subseteq O\}$, (ii) $j \in D^h(p^{t-1})$ and (iii) for all $k \in D^h(p^t)$ it holds that $p_k^t \leq p_j^{t-1}$. Since $p_k^t = p_k^{t-1} + 1$ for all $k \in O$ and $p_k^t = p_k^{t-1}$ for all $k \in N^t \setminus \{O\}$, it follows that $p_j^{t-1} = m^h$, otherwise j should still have been in the demand set of h at p^t. Hence $p_j^* = p_j^{t-1} = m^h$. This completes the proof. □

As an immediate consequence of the theorem, we have the following corollary.

Corollary 5.2 *The auction model with financially constrained bidders has at least one rationed equilibrium.*

The next result shows that the rationed equilibrium (p^*, π^*, R^*) generated by the auction has a particular and interesting property, namely when a bidder is rationed on some item j, then there exist prices $\widetilde{p}_k$ at most equal to the equilibrium prices p^*_k for all the other items $k \neq j$, such that at any price $\widetilde{p}_j$ greater than its equilibrium price p^*_j, the item j does not belong to the unconstrained demand set $D^h(\widetilde{p})$ for every $h \in M$, i.e. at $\widetilde{p}$ no bidder demands item j.

Proposition 5.3 *Let (p^*, π^*, R^*) be the rationed equilibrium generated by the auction. If $R^{i*}_j = 0$ for some i, then there exist $\widetilde{p}_k \leq p^*_k$ for all $k \neq j$, such that $j \notin D^h(\widetilde{p})$ for any $\widetilde{p}_j > p^*_j$ and any $h \in M$.*

Proof. Observe that there is only rationing for items assigned in Step 3. Let item j be assigned in Step 3 at stage t against price p^{t-1}_j. By construction of the auction we have that $p^{t-1}_k \leq p^*_k$ for all $k \in N^{t-1}$. Now, for all $k \neq j$, take $\widetilde{p}_k = p^{t-1}_k$ if $k \in N^{t-1} \setminus \{j\}$ and $\widetilde{p}_k = p^*_k$ if $k \in N \setminus N^{t-1}$. We now consider two cases.

Case A. Item j was overpriced at p^t. First consider a bidder $h \in M \setminus M^{t-1}$. Such a bidder has been assigned some item $\ell \in N \setminus N^{t-1}$ at some price system p^τ, $\tau < t-1$. It holds that $\ell \in D^h(p^\tau)$. Since $p^*_j = p^{t-1}_j \geq p^\tau_j$ and $p^*_\ell = p^\tau_\ell$, it follows that $j \notin D^h(\widetilde{p})$ for all $p_j > p^*_j = p^{t-1}_j$, whether or not j was in $D^h(p^\tau)$. Second, consider a bidder h in M^{t-1}. Since j was overpriced at p^t, bidder h did not demand j in stage t at price p^t. So, since in stage t only the items in N^t were available, we have that j does not belong to the demand set

$$D^h(p^t) = \{\ell \in N^t \cup \{0\} \ \mid \ p^t_\ell \leq m^i, \ V^i(\ell) - p^t_\ell = \max_{\{k \in N^t \cup \{0\} \ \mid \ p^t_k \leq m^i\}} (V^i(k) - p^t_k)\},$$

with respect to the set N^t of available items at stage t. Since $\widetilde{p}_k = p^{t-1}_k \leq p^t_k$ for $k \in N^{t-1} \setminus \{j\}$ and $\widetilde{p}_j \geq p^t_j = p^*_j + 1$, it follows that j does not belong to the demand set

$$D^h(\widetilde{p}) = \{\ell \in N \cup \{0\} \ \mid \ \widetilde{p}_\ell \leq m^i, \ V^i(\ell) - \widetilde{p}_\ell = \max_{\{k \in N \cup \{0\} \ \mid \ \widetilde{p}_k \leq m^i\}} (V^i(k) - \widetilde{p}_k)\},$$

with respect to the set of all items, irrespective of the prices of the items in $N \setminus N^t$.

Case B. Item j was an element of a minimal underdemanded set U^t with $|U^t| \geq 2$. Similarly as in Case A, for any bidder $i \in M \setminus M^{t-1}$ we have $j \notin D^i(\widetilde{p})$ for all $p_j > p^*_j = p^{t-1}_j$, whether or not j was in $D^i(p^\tau)$. Second, consider the bidders in M^{t-1} and let $T = \{i \in M^{t-1} \mid D^i(p^t) \cap U^t \neq \emptyset\}$ be the set of bidders that demand at p^t at least one item from U^t. According to Lemma 3.6, for any bidder in $h \in T$ we have that $|D^h(p^t) \cap U^t| \geq 2$. So, for any bidder $h \in M^{t-1}$ it holds that either $j \notin D^h(p^t)$, or $j, \ell \in D^h(p^t)$ for some

$\ell \neq j$. In the latter case such a bidder h is indifferent between j and ℓ at p^t. This implies that j will not be chosen by bidder $h \in T$ from the set N^t when $\widetilde{p}_j > p^*_j = p^{t-1}_j = p^t_j - 1$ and $\widetilde{p}_\ell = p^t_\ell - 1$. So, bidder $h \in M^{t-1}$ does not demand j at prices $\widetilde{p}_k$, $k \in N^{t-1} \setminus \{j\}$ and $\widetilde{p}_j > p^*_j$ when only the items in N^{t-1} are available. Analogously as in Case A, it follows that also $j \notin D^h(\widetilde{p})$, i.e., bidder $h \in M^{t-1}$ also does not have j is his demand set at $\widetilde{p}$ when all items in N available, irrespective of the prices $\widetilde{p}_k$ of the items k in $N \setminus N^{t-1}$. □

So far we have considered the case that some or all bidders may confront financial constraints. We have shown that the proposed ascending auction can handle such a situation and always finds a rationed equilibrium. One may naturally ask whether the proposed auction can find a Walrasian equilibrium when no bidder faces a budget constraint. The following theorem demonstrates that this is indeed the case. More specifically, if every bidder i is endowed with a sufficient amount m^i of money in the sense that $m^i \geq V^i(j)$ for all $j \in N$, then the ascending auction coincides with the DGS auction and finds a Walrasian equilibrium with the smallest equilibrium price vector in finitely many steps.

Theorem 5.4 *If $m^i \geq \max_{j \in N} \; V^i(j)$ for all $i \in M$, then the ascending auction coincides with the DGS auction and finds a Walrasian equilibrium with a minimal equilibrium price vector p^* in finitely many steps.*

Proof. It is sufficient to show that the ascending auction never generates an underdemanded set of items in any stage. It is true at stage 1 because the ascending auction starts with the reservation price vector C. Suppose that at some general stage t, there is no underdemanded set of items and O is the minimal overdemanded set of items chosen by the auctioneer as described in Step 2. We will show that there will be no underdemanded set of items at stage $t+1$. We first prove that no subset S of the set O is underdemanded at p^{t+1}. Because $m^i \geq \max_{j \in N} V^i(j)$ and $0 \notin O$, every bidder $i \in D^-_O(p^t)$ who demands items from S at p^t will continue to demand the same items in S and may demand other items as well at p^{t+1}. It follows from Lemma 3.2 that the set S cannot be underdemanded at p^{t+1}. Second, no subset S of $N^t \setminus O$ is underdemanded at p^{t+1}, because S is not underdemanded at p^t and the price of each item in $N^t \setminus O$ at stage $t+1$ is the same as at stage t and the price of each item in O is increased by one at stage $t+1$. Combining the two reasonings for the case $S \subseteq O$ and $S \subseteq N^T \setminus O$, it follows that also any $S \subseteq N^t$ with $S \cap O \neq \emptyset$ and $S \cap (N^t \setminus O) \neq \emptyset$ is underdemanded at p^{t+1}. So the ascending auction will never go to Step 3 and thus coincides exactly with the DGS auction. It is known that their auction finds an equilibrium with the minimal equilibrium price vector. □

Observe that when the condition of Theorem 5.4 holds, the auction never reaches Step 3 and thus $N^t = N$ in any stage t.

6 Conclusion

In this paper we investigated a general and practical market model in which an auctioneer wants to sell a number of items to a group of financially constrained bidders. Every bidder knows his values over the items and his budget privately and the auctioneer does not know this private information unless bidders tell her. When biddders face budget constraints, a Walrasian equilibrium typically fails to exist. Also the well-known ascending auction of Demange, Gale and Sotomayor (1986), as well as other auctions, might fail to allocate all the items. To overcome this the concept of rationed equilibrium has been proposed. Moreover and most importantly, an ascending (open) auction is developed which, starting with the seller's reservation price of each item, always ends up with a rationed equilibrium in finitely many steps. This makes the auction an attractive allocation mechanism in situations with financially constrained bidders. By using the auction the seller may increase his revenues and the total value in the market may increase compared to the disequilibrium situations in which not all items are allocated. Furthermore, because the auction yields a rationed equilibrium, it implies that a rationed equilibrium always exists in the assignment market with financial constraints. This result is parallel to the celebrated result obtained by Shapley and Shubik (1972), who proved the existence of a Walrasian equilibrium in the assignment market without financial constraints. In contrast to their method, our approach is constructive and gives us an exact solution. We have also shown that when no bidder is financially constrained, the proposed auction reduces to the auction of Demange, Gale and Sotomayor (1986).

References

[1] C. Azariadis and J. Stiglitz (1983), "Implicit contracts and fixed price equilibria," *Quarterly Journal of Economics,* 98, 1-22.

[2] J.P.Benoît and V. Krishna (2001), "Multiple-object auctions with budget constrained bidders," *Review of Economic Studies,* 68, 155-179.

[3] Y.K.Che and I. Gale (1996), "Expected revenue of the all-pay auctions and first-price sealed-bid auctions with budget constraints," *Economics Letters,* 50, 373-380.

[4] Y.K.Che and I. Gale (1998), "Standard auctions with financially constrained bidders," *Review of Economic Studies,* 65, 1-21.

[5] C. Cox (1980), "The enforcement of public price controls," *Journal of Political Economy*, 88, 887-916.

[6] V.P. Crawford and E.M. Knoer (1981), "Job matching with heterogeneous firms and workers," *Econometrica,* 49, 437-450.

[7] G. Demange, D. Gale, and M. Sotomayor (1986), "Multi-item auctions," *Journal of Political Economy,* 94, 863-872.

[8] J.H. Drèze (1975), "Existence of an exchange economy under price rigidities," *International Economic Review*, 16, 310-320.

[9] F. Gul and E. Stacchetti (1999), "The English auction with differentiated commodities," *Journal of Economic Theory*, 92, 66-95.

[10] P. Hall (1935), "On representitives of subsets," *Journal of London Mathematical Society*, 10, 26-30.

[11] G. Illing and U. Klüh (2003), *Spectrum Auctions and Competition in Telecommunications*, MIT Press, London.

[12] P. Klemperer (2004), *Auctions: Theory and Practice*, Princeton University Press, Princeton, New Jersey.

[13] T.C. Koopmans and M. Beckmann (1957), "Assignment problems and the location of economic activities," *Econometrica*, 25, 53-76.

[14] V. Krishna (2002), *Auction Theory*, Academic Press, New York.

[15] M. Kurz (1982), "Unemployment equilibria in an economy with linked prices," *Journal of Economic Theory*, 26, 110-123.

[16] G. van der Laan (1980), "Equilibrium under rigid prices with compensation for the consumers," *International Economic Review*, 21, 53-73.

[17] J.J. Laffont and J. Roberts (1996), "Optimal auction with financially constrained buyers," *Economics Letters*, 52, 181-186.

[18] E. Maskin (2000), "Auctions, development, and privatization: efficient auctions with liquidity-constrained buyers." *European Economic Review*, 44, 667-681.

[19] D. Mishra and D. Talman (2006), "Overdemand and underdemand in economies with indivisible goods and unit demands," DP No.2006-84, CentER for Economic Research, Tilburg University, Tilburg.

[20] D. Mishra and D. Veeramani (2006), "An ascending price procurement auction for multiple items with unit supply," *IEE Transactions*, 38, 127-140.

[21] T.R. Palfrey (1980), "Multiple object, discriminatory with bidding constraints: a game-theoretic analysis," *Management Science*, 26, 935-946.

[22] C. Pitchik and A. Schotter (1988), "Perfect equilibria in budget-constrained sequential auctions: an experimental study," *Rand Journal of Economics*, 19, 363-388.

[23] M. Rothkopf (1977), "Bidding in simultaneous auctions with a constraint on exposure," *Operations Research*, 25, 620-629.

[24] D. Salant (1997), "Up in the air: GTE's experience in the MTA auction for personal communications services licenses," *Journal of Economics and Management Strategy,* 6 (3), 549-572.

[25] L.S. Shapley and M. Shubik (1972), "The assignment game I: the core," *International Journal of Game Theory*, 1, 111-130.

[26] M. Sotomayor (2002), "A simultaneous descending bid auction for multiple items and unitary demand," *Revista Brasilerira de Economia Rio de Janeiro,* 56, 497-510.

[27] D. Talman and Z. Yang (2006), "A dynamic auction for differentiated items under price rigidities," DP No.2006-26, CentER for Economic Research, Tilburg University, Tilburg, forthcoming in *Economics Letters.*

[28] W. Vickrey (1961), "Counterspeculation, auctions, and competitive sealed tenders," *Journal of Finance*, 16, 8-37.