Weak Exemption, Weak Exclusion, and a characterization of the Reverse Talmud rule for bankruptcy problems

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Abstract

Although the principles of Exclusion and Exemption in bankruptcy problems are appealing, the specific conditions under which agents receive their claim, respectively nothing, seem arbitrary and no bankruptcy rule satisfies both. However, weakening these conditions by lowering the boundaries of what is considered a ‘small claim’, we characterize the Reverse Talmud rule by Weak Exemption, Weak Exclusion, Consistency and Weak Proportionality. Moreover, we show that it is the unique Self-Dual rule satisfying Weak Proportionality and either Weak Exemption or Weak Exclusion. Finally, we consider a more general class of rules containing the Constrained Equal Awards and Constrained Equal Losses rules.

Keywords: Bankruptcy problem, Exemption, Exclusion, Self-Duality, Reverse Talmud rule.

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1 Introduction

One of the most fundamental models in economic theory is that of the *bankruptcy* or *rationing* problem. This problem considers a set of agents that each have a certain nonnegative *claim* on a nonnegative *estate* such that the sum of the claims is smaller than the estate. The estate consists of a given amount of a single (perfectly divisible) good. The allocation problem then is how to divide the estate among the agents taking into account their claims. This bankruptcy problem models real life situations. Basic properties of an allocation rule is that all agents earn a nonnegative part of the estate and no agent gets more than its claim. An allocation rule that satisfies these two basic properties is called a *bankruptcy rule*.

The literature discusses many solutions (i.e. bankruptcy rules) to this problem, see e.g. Young (1987, 1988), Chun (1988), Dagan (1996), Herrero et al. (1999), Herrero and Villar (2001) and Thomson (2008), or O'Neill (1982), Aumann and Maschler (1985), Curiel et al. (1988), and Dagan and Volij (1993) for a game theoretic approach. Also see Thomson (1995) and Moulin (2001) for a survey.

The four most famous bankruptcy rules are the Proportional rule (that divides the estate proportionally to the agents claims), the Constrained Equal Awards rule (that divides the estate equally among the agents under the condition that nobody gets more than its claim), the Constrained Equal Losses rule (that divides the difference between the aggregate claim and the estate equally, provided no agent ends up with a negative transfer) and the Talmud rule (that combines the principles of the three rules above and that gives at most half of the claim to an agent when the Estate is small, at least half of the claim when the estate is big and is equal to the Proportional rule when the estate is equal to half of the sum of the claims). We refer to Herrero and Villar (2001) for a survey of these four rules. Most part of the literature is devoted to characterizing these and other bankruptcy rules by several appealing axioms. For a survey on this we refer to Thomson (1998). In this paper we introduce a new bankruptcy rule that has important properties in common with these four rules.

Although these rules have a long tradition in history, their axiomatizations have been developed in the last few decades. In Herrero and Villar (2001) the four rules are evaluated by their differences with respect to satisfying a selected set of axioms. It is known from the literature that a solution satisfies the properties of Equal Treatment of Equals, Scale Invariance, Composition, Path Independence and Consistency if and only if it is either the Proportional rule or the Constrained Equal Awards rule or the Constrained Equal Losses rule, see Moulin (2000). Further, Herrero and Villar (2001) show that from these three solutions the Constrained Equal Awards rule is the only one that satisfies Exemption, while the Constrained Equal Losses rule is the only one that satisfies Exclusion[1]. The Exemption property says that small claims are not held responsible for the shortages. In contrast, Exclusion ignores small claims. Although the principles of Exclusion and Exemption are appealing, the specific conditions under which an agent receives its claim, respectively nothing, seem arbitrary and are inconsistent in the sense that there is no bankruptcy rule that satisfies both. However, we can still respect the principles of Exclusion and Exemption but with weaker conditions under which an agent must receive its claim or nothing. Therefore we define weak Exemption and weak Exclusion and show that there exist solutions that satisfy both these properties. More precise, we show that there is a unique bankruptcy rule that satisfies these two properties, together with Consistency and Weak Proportionality (the last property saying that changing the estate always has an effect on the payoffs of agents with higher claims that is at least as much as the effect on payoffs of lower claim agents). This rule turns out to be the Reverse Talmud rule introduced in Chun, Schummer and Thomson (2001), see also Thomson (2008). This rule combines the principles of the Proportional-, CEA- and CEL rules and can be seen as some kind of counterpart of the Talmud rule. It gives the same solution as the Constrained Equal Losses rule when the estate is small compared to the sum of the claims, the same solution as the Constrained Equal Awards rule when the estate is large and it is again equal to the Proportional rule when the estate is equal to half of the sum of the claims.

An important property that this Reverse Talmud rule has in common with the Proportional rule, but is not satisfied by the Constrained Equal Awards rule nor the Constrained Equal Losses rule is Self-Duality, see Chun, Schummer and Thomson (2001). We show that the properties of Weak Exemption and Weak Exclusion are each others dual, and so the Reverse Talmud rule can also be characterized as the unique Self-Dual solution that satisfies Consistency, Weak Proportionality and either Weak Exemption or Weak Exclusion.

Finally, we will also show that when we parametrize Weak Exclusion and Weak Exemption, we obtain a class of bankruptcy rules, containing the Constrained Equal Awards and Constrained Equal Losses rules as its two extreme elements and containing the Reverse Talmud rule as some kind of compromise solution between these two rules. All solutions in this class are CIC-rules (first Constant, then Increasing, then Constant) as considered in Thomson (2008).

This paper is organized as follows. In Section 2 we discuss some preliminaries. In Section 3 we characterize the Reverse Talmud rule using Weak Exemption and Weak Exclusion. In Section 4 we discuss and axiomatize the general class of bankruptcy rules.

[1] In fact, Herrero and Villar (2001) characterize these solutions by Path Independence, Consistency and Exemption, respectively Composition, Consistency and Exclusion.

Finally, Section 5 contains some concludiong remarks.

2 Preliminaries

Let $\mathcal{N}$ be a finite (or countably infinite) set of agents. An **allocation situation** is given by a finite set $N \subset \mathcal{N}$ and an amount $E \geq 0$, refered to as the *estate*, of a certain good (money) to be distributed among the agents. An allocation situation is called a **bankruptcy situation** (or a **rationing problem** (see e.g. Moulin (2000)), when each agent $i \in N$ has a claim $c_i \geq 0$ on the good such that $\sum_{i \in N} c_i \geq E$. In the sequel we denote a bankruptcy situation by the triple (N, E, c), where $c = (c_i)_{i \in N}$ is the collection of claims. For given N, the collection of all bankruptcy situations (N, E, c) is denoted by $\mathcal{B}^N$. Further $\mathcal{B} = \cup_{N \subset \mathcal{N}} \mathcal{B}^N$ denotes the collection of all bankruptcy situations over all populations. For given $N \subset \mathcal{N}$ and any $S \subseteq N$, c_S denotes the collection of claims $(c_i)_{i \in S}$ restricted to S, and $c(S) = \sum_{i \in S} c_i$ is the sum of claims of agents in S. Further we denote the aggregate claim by $C = \sum_{i \in N} c_i$ and the aggregate loss by $L = C - E$. Note from $C = L + E$ and $E \geq 0$, that $L \leq C$, so that $(N, L, c) \in \mathcal{B}$ is the bankruptcy problem induced by (N, E, c) in which the aggregate loss L has to be distributed among the players. For given $N \subset \mathcal{N}$, an allocation $x \in \mathbb{R}^n$ assigns a payoff x_i to any $i \in N$. For $S \subseteq N$ and $x \in \mathbb{R}^n$, we also denote $x_S = (x_i)_{i \in S}$ and $x(S) = \sum_{i \in S} x_i$. A **bankruptcy rule** on $\mathcal{B}$ is a mapping F that assigns to every $(N, E, c) \in \mathcal{B}$ a unique allocation $F(N, E, c)$ that is **efficient** and **individually non-negative and claim bounded**, i.e.,

(i) $\sum_{i \in N} F_i(N, E, c) = E$,
(ii) $0 \leq F_i(N, E, c) \leq c_i$ for any $i \in N$.

So, a bankruptcy rule always distributes exactly the worth of the estate such that no agent gets less than zero or more than its claim. For given bankruptcy rule F, the **dual rule** F^* of F, see Aumann and Maschler (1985), is obtained by distributing the aggregate loss $L = C - E$ among the players according to F, i.e., $F_i^*(N, E, c) = c_i - F_i(N, L, c)$, $i \in N$. Note that F^* satisfies (i) and (ii), and thus also is a bankruptcy rule.

We now recall four bankruptcy rules on $\mathcal{B}$ with a long history of being applied in real life situations[2]. For any (N, E, c), the four rules are defined as follows.

- **Proportional rule P**: $P_i(N, E, c) = \frac{c_i}{C} E$, $i \in N$.

[2]In Herrero and Villar (2001) the first three are called the 'three musketeers', while the fourth rule, introduced formally by Aumann and Maschler (1985), plays the role of the most famous fourth musketeer D'Artagnan.

- **Constrained Equal Awards rule CEA**: Let λ^* be the solution to $\sum_{i \in N} \min[c_i, \lambda] = E$. Then $CEA_i(N, E, c) = \min[c_i, \lambda^*]$, $i \in N$.
- **Constrained Equal Losses rule CEL**: Let λ^* be the solution to $\sum_{i \in N} \max[0, c_i - \lambda] = E$. Then $CEL_i(N, E, c) = \max[0, c_i - \lambda^*]$, $i \in N$.
- **Talmud rule T**: For all $i \in N$,

$$T_i(N, E, c) = \begin{cases} CEA_i(N, E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2}C, \\ c_i - CEA_i(N, L, \frac{1}{2}c) & \text{if } E \geq \frac{1}{2}C. \end{cases}$$

The Proportional rule distributes E proportional to the claims; the CEA rule gives to any player the same payoff, but bounded from above by its claim; the CEL rule distributes the aggregate loss equal among the players, but no player gets a negative payoff; and finally the Talmud rule applies the CEA rule with half of the claims if the estate is at most half of the aggregate claim, otherwise each player first gets its claim and then the aggregate loss $L = C - E$ is distributed by applying the CEA rule with half of the claims. The Talmud rule is the extension of the so-called **Contested Garment** rule for two players, which can be found already in the Talmud, to more than two players.

Recently, Chun, Schummer and Thomson (2001) introduced the Reverse Talmud rule RT, which applies the CEL rule with half of the claims if the estate is at most half of the aggregate claim, otherwise each player first gets its claim and then the aggregate loss $L = C - E$ is distributed by applying the CEL rule with half of the claims.

- **Reverse Talmud rule RT**: For all $i \in N$,

$$RT_i(N, E, c) = \begin{cases} CEL_i(N, E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2}C, \\ c_i - CEL_i(N, L, \frac{1}{2}c) & \text{if } E \geq \frac{1}{2}C. \end{cases}$$

For the two-agent problem with claims $c_1 < c_2$, the RT rule is illustrated in Figure 1. The curve $pqrs$ in Figure 1 shows how the payoffs depend on E. When E increases from 0 to $C = c_1 + c_2$, the corresponding payoff vectors first move from $p = (0, 0)$ to $q = (0, \frac{c_2 - c_1}{2})$, then from q to $r = (c_1, \frac{c_1 + c_2}{2})$ and then from r to $s = (c_1, c_2)$.

Since the CEA and CEL rule are dual to each other, i.e. $CEL_i(N, E, c) = c_i - CEA_i(N, E, c)$, $i \in N$, the RT rule can also be written as

$$RT_i(N, E, c) = \begin{cases} \frac{1}{2}c_i - CEA_i(N, \frac{1}{2}C - E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2}C, \\ \frac{1}{2}c_i + CEA_i(N, E - \frac{1}{2}C, \frac{1}{2}c) & \text{if } E \geq \frac{1}{2}C, \end{cases} \tag{2.1}$$

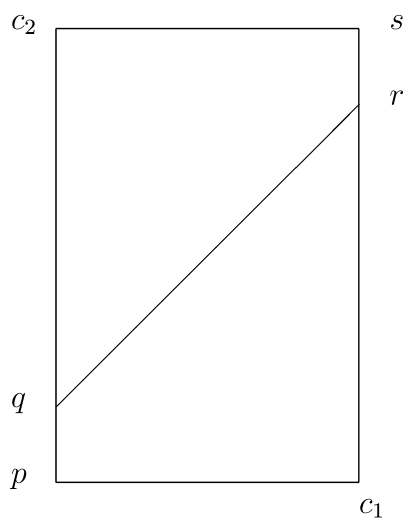

Figure 1: The RT rule for two-agent problems with $c_1 < c_2$

showing that the RT rule first assigns to each agent half of its claim and then distributes the remaining loss or surplus according to the CEA rule with half of the claims. Observe that analogously we obtain the Talmud rule when we distribute the remaining loss or surplus by applying the CEL rule with half of the claims instead of the CEA rule. From this it follows that the RT rule, like the Proportional rule and the Talmud rule, is Self-Dual, see also Chun, Schummer and Thomson (2001)[3]

Self-Duality SD: $F(N, E, c) = F^*(N, E, c)$.

The rules that we consider in this paper all satisfy the traditional property of Consistency.

Consistency C: For any $S \subset N$, $F_i(S, E^S, c_S) = F_i(N, E, c)$, $i \in S$, where $E^S = \sum_{i \in S} F_i(N, E, c)$.

In the literature several axiomatic characterizations of the Proportional, CEA, CEL and Talmud rule can be found. In this paper we are, in particular, interested in the so-called Exemption and Exclusion properties introduced by Herrero and Villar (2001). Exemption states that an agent whose claim is at most equal to the Estate divided by the number of agents earns it claim, while Exclusion states that an agent whose claim is at most equal to the aggregate loss divided by the number of agents earns nothing. Let $n = |N|$ denote the cardinality of N.

[3]In fact, Thomson (2008) shows that the RT rule is the only self-dual CIC-rule, see also Section 4.

	Reverse Talmud rule			Talmud rule		
	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$E = 100$	0	25	75	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$E = 200$	$16\frac{2}{3}$	$66\frac{2}{3}$	$116\frac{2}{3}$	50	75	75
$E = 300$	50	100	150	50	100	150
$E = 400$	$83\frac{1}{3}$	$133\frac{1}{3}$	$183\frac{1}{3}$	50	125	225
$E = 500$	100	175	225	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$

	CEA rule			CEL rule		
	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$E = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	0	0	100
$E = 200$	$66\frac{2}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$	0	50	150
$E = 300$	100	100	100	0	100	200
$E = 400$	100	150	150	$33\frac{1}{3}$	$133\frac{1}{3}$	$233\frac{1}{3}$
$E = 500$	100	200	200	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$

Table 1: T-, RT-, CEA- and CEL-rule applied to Example 2.1

Exemption Exe: For all $i \in N$, $F_i(N, E, c) = c_i$ if $c_i \leq \frac{1}{n}E$.

Exclusion Exc: For all $i \in N$, $F_i(N, E, c) = 0$ if $c_i \leq \frac{1}{n}L$.

In this paper we consider a class of bankruptcy rules that contains the CEA and CEL rule as extreme elements, and the RT rule somehow 'in the middle'. Within this class the CEA rule is the only one that satisfies Exemption, while the CEL rule is the only one that satisfies Exclusion.

We end these preliminaries with an example.

Example 2.1 We consider the well-known three-agent example from the Talmud with claims 100, 200 and 300 and compute the outcome according to the Reverse Talmud-, Talmud-, CEA- and CEL rules for estates verying from 100 to 500. The outcomes are given in the Table 1. In this table the rows correspond to different values of the Estate, while the columns correspond to the claims of the agents.

3 Weak Exemption and Weak Exclusion

Although the two principles of Exemption and Exclusion are appealing, the specific bounds under which they hold seem rather arbitrary and have some problems. Both principles are applied in real life bankruptcy situations. The Exemption property says that small claims are not held responsible for the shortages. When an agent has a claim that is smaller than what should be received when the estate E is equally divided between the claimants, then the agent should be granted its full claim. From the viewpoint of distributing aggregate losses, i.e., the agents are first awarded their full claim and then are taxed to pay for the aggregate loss, Exemption reflects the general principle of progressive taxation: agents with small claims don't have to contribute in sharing the aggregate loss. In contrast, Exclusion ignores small claims. When an agent has a claim that is smaller than the average loss $\frac{1}{n}L$, then the claim is disregarded and the agent gets nothing. From the viewpoint of distributing aggregate losses, Exclusion reflects the general principle of degressive taxation: small claims are fully taxed away, whereas bigger claims are partially granted.

Although both principles seem appealing, a main problem is that there is no rule that can satisfy the principles of Exemption and Exclusion simultaneously. Clearly, for instance when $C = 2E$, we have $L = E$ and for $c_i \leq \frac{1}{n}E = \frac{1}{n}L$ the two principles require simultaneously that $F_i(N, E, c) = c_i$ and $F_i(N, E, c) = 0$, which is contradictory. However, this is not due to the principles of Exemption and Exclusion, but is caused by what are considered to be 'small' claims in relation to the size of the estate.

The main motivation of this paper is to give a rule that simultaneously satisfies a type of Exemption and a type of Exclusion property. This rule has the property that small claims are fully awarded when the estate is large, whereas small claims are fully ignored when the estate is small. We call these properties **Weak Exemption** and **Weak Exclusion**. They say that the claim of agent i is fully awarded when $c_i \leq \frac{E-L}{n}$, while the claim is fully ignored when $c_i \leq \frac{L-E}{n}$. Observe that $E - L \geq 0$ if and only if $E \geq \frac{1}{2}C$.

11. **Weak Exemption W-Exe**: For all $i \in N$, $F_i(N, E, c) = c_i$ if $c_i \leq \frac{E-L}{n}$.

12. **Weak Exclusion W-Exc**: For all $i \in N$, $F_i(N, E, c) = 0$ if $c_i \leq \frac{L-E}{n}$.

Since $E - L \leq E$ and $L - E \leq L$, the range of claims for which Weak Exemption (respectively Weak Exclusion) requires the agent to be allocated its claim (respectively zero) is smaller then those for which this is required under Exemption (respectively Exclusion). Thus, Exemption (respectively Exclusion) implies Weak Exemption (respectively Weak Exclusion). Observe that when $E = L$, the two properties do not require anything.

Note that none of the four classical bankruptcy rules (P, CEA, CEL and T) satisfy both Weak Exemption and Weak Exclusion. However, it turns out that the RT rule satisfies these two properties.

Lemma 3.1 *The Reverse Talmud rule on $\mathcal{B}$ satisfies Weak Exemption and Weak Exclusion.*

Proof. First, consider the case that $E - L \geq 0$ and thus $E \geq \frac{1}{2}C$. Then $RT_i(N,E,c) = \frac{1}{2}c_i + CEA_i(N, E - \frac{1}{2}C, \frac{1}{2}c)$. If $c_i \leq \frac{E-L}{n} = \frac{2E-C}{n}$, then $\frac{c_i}{2} \leq \frac{E-\frac{1}{2}C}{n}$ and we have that $CEA_i(N, E - \frac{1}{2}C, \frac{1}{2}c) = \frac{1}{2}c_i$. Hence $RT_i(N,E,c) = c_i$.

Similarly, when $E - L \leq 0$ and thus $E \leq \frac{1}{2}C$, we have that $RT_i(N,E,c) = \frac{1}{2}c_i - CEA_i(N, \frac{1}{2}C - E, \frac{1}{2}c)$. If $c_i \leq \frac{L-E}{n} = \frac{C-2E}{n}$, then $\frac{c_i}{2} \leq \frac{\frac{1}{2}C-E}{n}$ and thus $CEA_i(N, \frac{1}{2}C - E, \frac{1}{2}c) = \frac{1}{2}c_i$. Hence $RT_i(N,E,c) = 0$. □

To give a full characterization of the Reverse Talmud rule, we now state one more property, to be called **Weak Proportionality** which says that when the estate becomes larger, the increase in the payoffs of the players is nondecreasing in the size of the claims: a player with a bigger claim receives at least the same of the additional amount of the estate as a player with a smaller claim.

13. **Weak Proportionality WP**: For $i, j \in N$ with $c_i \geq c_j$ it holds that $F_i(N,E',c) - F_i(N,E,c) \geq F_j(N,E',c) - F_j(N,E,c)$ for every E and E' such that $E' > E$.

Note that WP of a bankruptcy rule implies *Claim Monotonicity* saying that agents with higher claims get at least as much than agents with smaller claims, i.e. $F_i(N,E,c) \geq F_j(N,E,c)$ if $c_i \geq c_j$.

Lemma 3.2 *The Reverse Talmud rule on $\mathcal{B}$ satisfies Weak Proportionality.*

Proof. The RT solution first awards each agent half of its claim and then the remaining Loss or Surplus $|E - \frac{1}{2}C|$ is distributed according to the CEA rule with half of the claims. It is easy to verify that the CEA rule satisfies Weak Proportionality. Indeed, two players i and j receive the same additional payoff when the claim increases from E to E' if both still do not obtain their full payoff at E'. Otherwise the agent with the bigger claim gets at least the same additional payoff as the agent with the smaller claim.

We now consider three cases for the RT rule. For two agents i and j, suppose that $c_i \geq c_j$. When both E and E' are smaller than half of the aggregate claim C, then it follows from the Weak Proportionality of the CEA rule that agent i faces at least the same additional loss as agent j when the estate decreases from E' to E. Similarly, when both E and E' are larger than half of the aggregate claim, then agent i faces at least the same additional payoff as agent j when the estate increases from E to E'. Finally, when E is at most half of the aggregate claim and E' is at least half of the aggregate claim, then, first, agent i faces at least the same additional loss as agent j when the estate decreases from

$\frac{1}{2}C$ to E and, second, agent i receives at least the same additional payoff as agent j when the estate increases from $\frac{1}{2}C$ to E'. □

We now come to the main theorem which characterizes the RT rule.

Theorem 3.3 *The Reverse Talmud rule is the unique rule on $\mathcal{B}$ that satisfies Weak Exemption, Weak Exclusion, Consistency and Weak Proportionality.*

Proof. From the Lemma's 3.1 and 3.2 we have that the RT rule satisfies Weak Exemption and Weak Exclusion. Consistency is shown in Thomson (2008). It remains to show that the four properties uniquely determine the outcome.

Let $N \subset \mathcal{N}$. Without loss of generality we number the agents in N from 1 to n, i.e. $N = \{1, \ldots, n\}$, and withhout loss of generality we assume that $c_1 \leq c_2 \leq \ldots \leq c_n$. Let $x \in \mathbb{R}^n$ denote the vector of payoffs assigned by the RT rule. For $n = 1$, by definition the only rule is to assign E fully to the single player in N, i.e., $x_1 = E$. Next we consider $n = 2$. Define $E^0 = -\frac{1}{2}c_1 + \frac{1}{2}c_2$ and $E^1 = \frac{3}{2}c_1 + \frac{1}{2}c_2$. Observe $E^0 \geq 0$ and $E^1 \leq C$. We distinguish three cases. First consider the case that $E \leq E^0$. This yields $c_1 \leq c_2 - 2E$ and thus $2c_1 \leq c_1 + c_2 - 2E = C - 2E = L - E$. Hence Weak Exclusion implies that $x_1 = 0$, $x_2 = E$ and thus the payoffs are uniquely determined. The second case is $E \geq E_1$. This yields $3c_1 \leq 2E - c_2$ and thus $2c_1 \leq 2E - (c_1 + c_2) = 2E - C = E - L$. Hence Weak Exemption implies that $x_1 = c_1$, $x_2 = E - c_1$ and again the payoffs are uniquely determined. It remains to consider the case $E_0 \leq E \leq E_1$. We have seen that $x_1^0 = 0,\ x_2^0 = E^0$ are the payoffs at $E = E^0$ and $x_1^1 = c_1,\ x_2^1 = E^1 - c_1$ are the payoffs at $E = E^1$. So, when E goes from E^0 to E^1, the payoff of agent 1 increases from $x_1^0 = 0$ to $x_1^1 = c_1$ and the payoff of agent 2 increases from $x_2^0 = E^0 = -\frac{1}{2}c_1 + \frac{1}{2}c_2$ to $x_2^1 = E^1 - c_1 = \frac{1}{2}c_1 + \frac{1}{2}c_2$. So, when E goes from E^0 to E^1, the increase in both payoffs is equal to c_1. Weak Proportionality now requires that for any E between E^0 and E^1, $x_1 = x_1^0 + \frac{1}{2}(E - E^0)$ and $x_2 = x_2^0 + \frac{1}{2}(E - E^0)$. Hence for any E the payoffs are uniquely determined when $n = 2$.

For $n \geq 2$, we now proceed by induction. Suppose that the payoffs are uniquely determined when the number of agents is at most $n - 1$. Define $E^0 = \frac{C}{2} - \frac{nc_1}{2} \geq 0$ and $E^1 = \frac{C}{2} + \frac{nc_1}{2} \leq C$. We distinguish three similar cases as for $n = 2$. First consider the case that $E \leq E^0 = \frac{C}{2} - \frac{nc_1}{2}$. This yields $nc_1 \leq C - 2E = L - E$ and Weak Exclusion implies that $x_1 = 0$ and so that $\sum_{i=2}^n F_i(N, E, c) = E$. Then $x_2, \ldots, x_n$ follow from Consistency and the induction hypothesis (i.e. the four properties uniquely determine the payoffs when the number of agents is equal to $n - 1$.). Hence the payoffs are uniquely determined. The second case is $E \geq E_1$. This yields $nc_1 \leq 2E - C = E - L$ and Weak Exemption implies that $x_1 = c_1$ and so $\sum_{i=2}^n F_i(N, E, c) = E - c_1$. Again $x_2, \ldots, x_n$ are uniquely determined by Consistency and the induction hypothesis. It remains to consider the case $E_0 \leq E \leq E_1$. Let $x^0 \in \mathbb{R}^n$ be the uniquely determined vector of payoffs at E^0 and $x^1 \in \mathbb{R}^n$ at E^1. It holds that $x_1^0 = 0$ and $x_1^1 = c_1$. Further $E^1 - E^0 = nc_1$. So, when E goes from E^0 to E^1, the total payoff increases with nc_1, whereas the payoff of the smallest claim increases from 0 to c_1. Weak Proportionality now requires that the change in payoff for all agents should be the same, and thus for any E between E^0 and E^1, $x_i = x_i^0 + \frac{1}{n}(E - E^0)$ for all $i \in N$. Hence for any E the payoffs are uniquely determined. □

Realizing that Weak Exemption and Weak Exclusion are *dual* to each other, one obtains immediately another characterization of the RT-rule. A property P^* is the dual of a property P if for any rule F: F satisfies property P if and only if its dual F^* satisfies property P^*.

Lemma 3.4 *The properties of Weak Exemption and Weak Exclusion are dual to each other.*

Proof. Let F be a bankruptcy rule. Then [F^* satisfies Weak Exclusion] if and only if [$c_i \leq \frac{L-E}{n} \Rightarrow F_i^*(N, E, c) = 0$] if and only if [$c_i \leq \frac{L-E}{n} \Rightarrow c_i - F_i(N, L, c) = 0$] if and only if [$c_i \leq \frac{L-E}{n} \Rightarrow F_i(N, L, c) = c_i$] if and only if [$F$ satisfies Weak Excemption]. For the latter assertion, recall that $E + L = C$. So when applying the bankruptcy rule F to L we have that the loss is given by $E = C - L$ and the condition for Weak Excemption becomes that $c_i \leq \frac{L-E}{n}$. □

Herrero and Villar (2001) already mention that the properties of Exemption and Exclusion are each other's dual. Since there does not exist a rule that satisfies both, it follows that there does not exist a rule F that is Self-Dual and satisfies Exemption. Also, there does not exist a rule F that is Self-Dual and satisfies Exclusion. Contrary to Exemption and Exclusion, we have seen above that there do exist solutions that satisfy both weak versions of these properties, such as the Reverse Talmud rule. Since F being Self-Dual and satisfying a property P, implies that it also satisfies property P^*, the next corollary follows immediately from Theorem 3.3, Lemma 3.4 and the Reverse Talmud rule satisfying Self-Duality.

Corollary 3.5 *The Reverse Talmud rule is the unique rule on $\mathcal{B}$ that satisfies Weak Exemption, Self-Duality, Consistency and Weak Proportionality. Also the Reverse Talmud rule is the unique rule on $\mathcal{B}$ that satisfies Weak Exclusion, Self-Duality, Consistency and Weak Proportionality.*

4 A class of bankruptcy rules

Moreno-Ternero and Villar (2006) generalize the Talmud rule to a class of bankruptcy rules that contains the CEA rule and the CEL rule as special (extreme) cases. In this section we generalize the RT rule to a class of rules by parametrizing the properties of Exemption and Exclusion for $\alpha, \beta \in [0,1]$. Also this class contains the CEA rule and the CEL rule as two extreme cases.

11. α-**Exemption** α-**Exe**: For all $i \in N$, $F_i(N,E,c) = c_i$ if $\alpha c_i \leq \frac{E-(1-\alpha)C}{n}$.

12. β-**Exclusion** β-**Exc**: For all $i \in N$, $F_i(N,E,c) = 0$ if $\beta c_i \leq \frac{\beta C - E}{n}$.

Observe that for $\alpha = \beta = 1$ we have Exemption and Exclusion and that for $\alpha = \beta = \frac{1}{2}$ we have Weak Exemption and Weak Exclusion. As we remarked before, a bankruptcy rule cannot satisfy both properties for $\alpha = \beta = 1$, but the RT rule satisfies both properties for $\alpha = \beta = \frac{1}{2}$. In fact, one can verify that there always exist bankruptcy rules that satisfy α-Exe and β-Exc if $\alpha + \beta = 1$, i.e. if $\beta = 1 - \alpha$. Similar as the proof of Theorem 3.3 one can show that there always is a unique bankruptcy rule that satisfies these properties together with Consistency and Weak Proportionality.

Theorem 4.1 *For any $\alpha \in [0,1]$ there is a unique rule on $\mathcal{B}$ that satisfies $(1-\alpha)$-Exemption, α-Exclusion, Consistency and Weak Proportionality. For all $i \in N$, this rule is given by*

$$T_i^{\alpha}(N,E,c) = \begin{cases} CEL_i(N,E,\alpha c) & \text{if } E \leq \alpha C, \\ c_i - CEL_i(N,L,(1-\alpha)c) & \text{if } E \geq \alpha C. \end{cases}$$

Proof. The proof of this theorem goes along similar lines as the the proofs of corresponding results in the previous section and is therefore omitted. □

Clearly, the class described in Theorem 4.1 contains the CEA rule (for $\alpha = 0$), the CEL rule (for $\alpha = 1$) and the RT rule (for $\alpha = \frac{1}{2}$). Note also that the RT rule is the unique Self-Dual rule in this class. Also note that this class is closed under Self-Duality in the sense that if a rule F belongs to this class, then also its dual rule belongs to this class. More precisely, the dual rule of T^{α} is $T^{1-\alpha}$, $\alpha \in [0,1]$. Further, we remark that all these rules are CIC rules as considered in Thomson (2008). According to these rules, when the Estate increases starting at zero, the payoff of any agent i first is constant (at zero) upto some Estate level a_i (possibly 0) then increases upto the claim level c_i and from there is constant at the claim level. Moreover, if for some agents the payoff increases when the Estate goes from E to E' then the payoffs for all these agents increase by the same amount. Obviously, if there are two agents then the class of CIC rules and the class of T^{α} rules coincide since by α-Exclusion the payoff of the small claimant, say claimant 1, is constant at zero upto

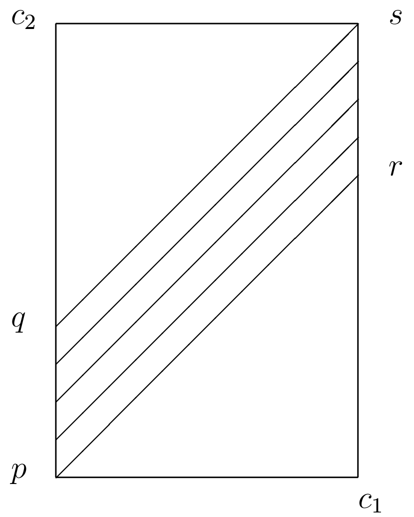

Figure 2: The class T^α for two-agent problems with $c_1 < c_2$

Estate level $\alpha(c_2 - c_1)$, while by $(1-\alpha)$-Exemption it must be equal to the claim c_1 if the Estate is more than $\alpha(c_2 - c_1) + 2c_1$. But this implies that between these two Estate levels the payoffs of both agents increase by the same amount. However, if there are more than two agents, then not all CIC rules are T^α rules.

For two agent problems the class is illustrated in Figure 2. The curve prs shows how the payoffs depend on E for the CEA rule and the curve pqs for the CEL rule. For $0 < \alpha < 1$, the curve of payoff vectors first goes from p in the direction of q, then somewhere between p and q it moves from the boundary $x_1 = 0$ upwards under 45 degrees to the boundary $x_1 = c_1$ and then it moves to the point s.

Besides the properties mentioned in Theorem 4.1 the rules T^α, $\alpha \in [0,1]$ are parametric[4] and satisfy other traditional properties such as Equal Treatment of Eequals[5], Continuity[6,7] (see Thomson (2008)) and Scale Invariance[8].

[4]A bankruptcy rule is parametric if there exists a function $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $[a,b] \subset \mathbb{R} \cup \{-\infty, +\infty\}$, such that f is continuous and weakly monotonic in its first argument, such that (i) $F_i(N,E,c) = f(\lambda, c_i)$ for all $(N,E,c) \in \mathcal{B}$ and for some $\lambda \in [a,b]$, (ii) $f(a,x) = 0$, for all $x \in \mathbb{R}_+$, and (iii) $f(b,x) = x$, for all $x \in \mathbb{R}_+$.

[5]A bankruptcy rule satisfies Equal Treatment of Equals if for all $i,j \in N$, $F_i(N,E,c) = F_j(N,E,c)$ if $c_i = c_j$.

[6]A bankruptcy rule satisfies Continuity (on E) ifn for all sequences of problems (N,E^l,c) such that $E^l \to E$, it holds that $F(N,E^l,c) \to F(N,E,c)$.

[7]Note that from Young (1987) it follows that a bankruptcy rule satisfies Equal Treatment of Equals, Continuity and Consistency if and only if it is Parametric.

[8]A bankruptcy rule satisfies Scale Invariance if for all $\lambda > 0$, $F_i(N, \lambda E, \lambda c) = \lambda F_i(N,E,c)$, $i \in N$.

5 Concluding remarks

The properties of Exemption (satisfied by the CEA rule) and Exclusion (satisfied by the CEL rule) for bankruptcy rules have as major disadvantage that there is no bankruptcy rule that satisfies both properties, although the principles behind both properties seem reasonable. But the bounds on the claims under which these properties require that an agent gets its claim, respectively gets zero, seem rather arbitrary. Therefore we weakened these bounds in a very natural way by replacing the worth of the Estate (respectively Loss) by the difference between the Estate and the Loss (respectively the difference between the Loss and the Estate). It turned out that there is a unique bankruptcy rule satisfying these weaker properties together with Consistency and Weak Proportionality which is the Reverse Talmud rule. Moreover, this rule is characterized by Self-Duality, Consistency, Weak Proportionality and either Weak Exemption or Weak Exclusion. We also generalized this Reverse Talmud rule by parametrizing the properties of Weak Exemption and Weak Exclusion, obtaining a class of bankruptcy rules that contains the CEA rule and the CEL rule as extreme cases, and are all CIC rules as considered in Thomson (2008).

In order to compare the new rules with the four classic rules (CEA, CEL, Proportional and Talmud rule) we conclude by recalling some known characterizations as summarized by Herrero and Villar (2001). Moulin (2000) showed that a rule satisfies the axioms ETE, SI, C, Composition (Comp)[9] and Path independence (PI)[10] if and only if it is the P, CEA or CEL bankruptcy rule. It follows from Young (1988) that P is the unique rule satisfying ETE, Comp, C and SD.[11] Besides the characterization of the CEA rule by PI, C and Exe, it follows from Dagan (1996) that the CEA rule is also the unique rule satisfying ETE, Comp, C and Independence of Claims Truncation (ICT).[12] Besides being the only rule satisfying Comp, C and Excl, it follows from Dagan (1996) that the CEL rule is characterized by ETE, PI, Cons and Composition from Minimal Rights (CMR).[13] Finally, the Talmud rule is characterized by C, ICT and CMR. It is also characterized by C, CMR and SD. From the properties mentioned above, the RT rule does not satisfy

[9]A bankruptcy rule satisfies Composition if $F_i(N, E_1+E_2, c) = F_i(N, E_1, c)+F_i(N, E_2, c-F(N, E_1, c))$ for all $i \in N$, $E_1,\ E_2 \geq 0$ and $C \geq E_1 + E_2$.

[10]A bankruptcy rule satisfies Path Independence if $F_i(N, E_1, c) = F_i(N, E_1, F(N, E_1 + E_2, c))$ for all $i \in N$, $E_1,\ E_2 \geq 0$ and $C \geq E_1 + E_2$.

[11]Young (1988) showed that on a fixed set of agents P is characterized by ETE, Comp and SD.

[12]A bankruptcy rule satisfies Independence of Claims Truncation if $F_i(N, E, \widetilde{c}) = F_i(N, E, c)$, where $\widetilde{c}_i = \min[c_i, E]$, for all $i \in N$. Dagan (1996) shows that on a fixed set of agents CEA is characterized by ETE, Comp and ICT.

[13]A bankruptcy rule satisfies Composition from Minimal Rights if $F_i(N, E, c) = m_i(N, E, c)+F_i(N, \overline{E}, \overline{c})$, where $m_i(N, E, c) = \max[0, E-\sum_{j\neq i} c_j]$, $i \in N$, $\overline{E} = E-\sum_{i\in N} m_i(N, E, c)$ and $\overline{c}_i = c_i-m_i(N, E, c)$, $i \in N$. Dagan (1996) shows that on a fixed set of agents CEL is characterized by ETE, PI and CMR.

Comp, PI, ICT and CMR.

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