Comment on ‘Sharing a polluted river’

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Abstract In Ni and Wang (2007) two solutions are given to determine the division of the cost of cleaning a river among the agents located along the river. For each solution the authors give three different motivations, namely (i) an axiomatic characterization, (ii) the solution is the Shapley value of an appropriately defined cooperative TU-game, and (iii) the solution belongs to the Core of the associated game. In this comment we argue that the results (ii) and (iii) follow directly from more general results on inessential and auction or dual airport games. We also provide some alternative axiomatizations without the Additivity axiom. Finally, we state more general results for these cost allocation methods that follow immediately from well-known results on cooperative games.

Keywords: Fair allocation of pollution costs, cooperative game, Shapley value, Core.

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1 Introduction

Ni and Wang (2007) consider river *pollution cost sharing problems* (N, C), where $N = \{1, \ldots n\}$ is a finite set of agents who are located along a river from upstream to downstream, and $C = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$ is a pollution cost vector with c_i the cost incurred by agent $i \in N$ for cleaning the river. A *solution* to a pollution cost sharing problem (N, C) is a vector $x \in \mathbb{R}^n_+$ where x_i is the cost to be paid by agent $i \in N$ in the total joint cleaning cost $\sum_{j=1}^{n} c_j$. For fixed N, a *method* is a mapping that assigns a solution to each

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pollution cost vector C. The two methods proposed in Ni and Wang (2007) are the *Local Responsibility Sharing method* given by

$$x_i^{LRS}(C) = c_i, \ i = 1, \dots, n,$$

and the *Upstream Equal Sharing method* given by

$$x_i^{UES}(C) = \sum_{j=i}^{n} \frac{c_j}{j}, \ i = 1, \dots, n.$$

Further, two cooperative TU-games associated with a pollution cost sharing problem are defined, namely the so-called LR polluted river game (N, v^C) given by

$$v^C(S) = \sum_{i \in S} c_i, \ S \subseteq N,$$

and the DR polluted river game (N, w^C) given by

$$w^C(S) = \sum_{i=\min S}^{n} c_i, \ S \subseteq N,$$

where the players in N correspond to the agents along the river. Axiomatizations of these two methods are given using the following properties.

Additivity For any $C^1, \ C^2 \in \mathbb{R}_+^n$, we have $x(C^1 + C^2) = x(C^1) + x(C^2)$.

Efficiency $\sum_{i=1}^n x_i = \sum_{i=1}^n c_i$ for any $C \in \mathbb{R}_+^n$.

No Blind Cost For any $i \in N$ and $C \in \mathbb{R}_+^n$ such that $c_i = 0$, we have $x_i(C) = 0$.

Upstream Symmetry For any $i \in N$ and $C \in \mathbb{R}_+^n$ it holds that $x_j(C) = x_k(C)$ for all $j, k \leq i$, whenever $c_h = 0$ for all $h \in N \setminus \{i\}$.

Independence of Upstream Costs For any $i \in N$ and $C, C' \in \mathbb{R}_+^n$ such that $c_h = c'_h$ for all $h > i$, we have that $x_j(C) = x_j(C')$ for all $j > i$.

We now summarize the results of Ni and Wang (2007), where $\varphi(v)$ denotes the Shapley value of game v (Shapley, 1953) and $Core(v)$ the Core of game v (Gillies, 1953).

Proposition 1 For all $C \in \mathbb{R}_+^n$, we have $x_i^{LRS}(C) = \varphi(v^C)$.

Proposition 2 For all $C \in \mathbb{R}_+^n$, we have $\varphi(v^C) \in Core(v^C)$.

Theorem 1 The LRS method is the only method satisfying Additivity, Efficiency and No Blind Cost.

Proposition 3 For all $C \in \mathbb{R}^n_+$, we have $x_i^{UES}(C) = \varphi(w^C)$.

Proposition 4 For all $C \in \mathbb{R}^n_+$, we have $\varphi(w^C) \in Core(w^C)$.

Theorem 2 The UES method is the only method satisfying Additivity, Efficiency, Upstream Symmetry and Independence of Upstream Costs.

In the next section we show that the Propositions 1-4 follow directly from known results on inessential and auction (or dual airport) games. We also give some additional axiomatizations and some more general results that follow from the literature. To do so, we denote the class of all TU-games $v \colon 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ by $\mathcal{G}^N$, and recall some well-known game-theoretic properties of a solution f for cooperative games, i.e. f is a function on $\mathcal{G}^N$ that assigns a vector $f(v) \in \mathbb{R}^n$ to any game in $\mathcal{G}^N$. To avoid confusion with the properties above for cost allocation methods, we explicitly add the word 'game' to the first two properties for solutions.
-*Game-Additivity*: $f(v + w) = f(v) + f(w)$ for any $v, w \in \mathcal{G}^N$, where $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$;
-*Game-Efficiency*: $\sum_{i=1}^n f_i(v) = v(N)$ for any $v \in \mathcal{G}^N$;
-*Symmetry*: $f_i(v) = f_j(v)$ whenever $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$;
-*Strong Monotonicity*: $f_i(v) \geq f_i(w)$ whenever $v(S) - v(S \setminus \{i\}) \geq w(S) - w(S \setminus \{i\})$ for all $S \subseteq N$;
-*Zero-Independence*: for any $v \in \mathcal{G}^N$ and $k \in \mathbb{R}^n$, we have that $f_i(v + k) = f_i(v) + k_i$, $i \in N$, where $(v + k)(S) = v(S) + \sum_{i \in S} k_i$ for all $S \subseteq N$.

2 Results

2.1 The LRS and UES methods reconsidered

As a first observation with respect to the LRS method it is obvious that for any $C \in \mathbb{R}^n_+$, the game v^C is an *additive* or *inessential* game. A TU-game v is inessential if there exist numbers $\alpha_i \in \mathbb{R}$, $i \in N$, such that $v(S) = \sum_{i \in S} \alpha_i$ for all $S \subseteq N$. In the following, let v^α denote the inessential TU-game for numbers $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Clearly, any LR-polluted river game v^C is an inessential game with $\alpha_i = c_i$, $i \in N$, and, reversely, any inessential game with nonnegative worths can be obtained as a LR-polluted river game v^C. The second observation is that every Zero-Independent (game) solution assigns to each player in an inessential game v^α its individual singleton worth α_i. Finally it is obvious that $x = \alpha$ is the unique allocation in the Core of the inessential game v^α. This gives without further proof the following corollary, where $\mathcal{I}^N_+$ is the class of nonnegative inessential TU-games on N, i.e. inessential games v^α with $\alpha_i \geq 0$ for all $i \in N$.

Theorem 2.1 (i) *Let $v \in \mathcal{G}^N$. Then there exists $C \in \mathbb{R}^n_+$ such that $v = v^C$ if and only if $v \in \mathcal{I}^n_+$.*

(ii) *For any Zero-Independent solution f and any inessential game $v = v^\alpha$ it holds that $f_i(v) = v(\{i\}) = \alpha_i$ for all $i \in N$.*

(iii) *For any inessential game $v = v^\alpha$, it holds that $Core(v^\alpha) = \{\alpha\}$.*

The theorem implies the Propositions 1 and 2 of Ni and Wang (2007). From statement (i) it follows that $v^C \in \mathcal{I}^N_+$. Since the Shapley value is Zero-Independent, Proposition 1 then follows immediately from statement (ii). The statement even implies the more general result that the LRS method also coincides with any Zero-Independent solution, such as the nucleolus (Schmeidler (1969)) or the τ-value (Tijs (1981)). Further, the statements (ii) and (iii) imply Proposition 2.

Although the UES method looks more advanced than the LRS method, we argue that also the Propositions 3 and 4 follow immediately from well-known results from the literature on so-called auction or airport games. In fact, the DR-polluted river game w^C is the dual airport game corresponding to the airport situation with costs $\sum_{j=i}^n c_j$ for player (airplane) $i \in N$, see Littlechild and Owen (1973). Also, w^C is the auction game corresponding to the auctioning situation with $\sum_{j=i}^n c_j$ being the valuation of agent i for the object that is auctioned, see Graham, Marshall and Richard (1990). Let $\mathcal{A}^n$ be the class of auction games (which is equivalent to the class of dual airport games), i.e. a game w belongs to $\mathcal{A}^N$ if and only if there is an $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$ such that $w = w^\alpha$ with $w^\alpha(S) = \sum_{i=\min S}^n \alpha_i$, $S \subseteq N$. The next theorem is given without further proof. In fact, statement (i) is obvious and the statements (ii) and (iii) are well-known results on the class $\mathcal{A}^N$. Recall that $\varphi(v)$ denotes the Shapley value of v.

Theorem 2.2 (i) *Let $v \in \mathcal{G}^N$. Then there exists $C \in \mathbb{R}^n_+$ such that $v = w^C$ if and only if $v \in \mathcal{A}^n$.*

(ii) *(Littlechild and Owen (1973), Graham, Marshall and Richard (1990)) For any $w^\alpha \in \mathcal{A}^n$ it holds that $\varphi_i(w^\alpha) = \sum_{j=i}^n \frac{\alpha_j}{j}$, $i \in N$.*

(iii) *If $w^\alpha \in \mathcal{A}^n$, then $\varphi(w^\alpha) \in Core(w^\alpha)$.*

The theorem implies the Propositions 3 and 4 of Ni and Wang (2007). Taking $\alpha = C$, Proposition 3 follows from (i) and (ii) and Proposition 4 from (i) and (iii).

Since the UES method can be obtained as the Shapley value of an auction or dual airport game, it is also similar to the *serial cost sharing mechanism* for cost sharing problems as considered in Moulin and Shenker (1992)[3].

[3]To be specific, the serial cost sharing mechanism is obtained as the Shapley value of the auction game where the valuation of agent i equals the cost of producing the sum of demands of all agents, where agent

2.2 Axiomatizations without Additivity

We now want to provide alternatives for the characterizations of the LRS and UES methods given in Theorems 1 and 2 without the Additivity axiom. First we consider the UES method. Young (1984) characterized the Shapley value on the class $\mathcal{G}^N$ of all TU-games by Game-Efficiency, Symmetry and Strong Monotonicity, but without using Game-Additivity. In van den Brink (2004) it is shown that these axioms also characterize the Shapley value on the class of auction games. Since any cost allocation method given by $x(C) = f(w^C)$ satisfies Independence of Upstream Costs when f satisfies Strong Monotonicity[4], Independence of Upstream Costs of the UES method follows from $x^{UES}(C) = \varphi(w^C)$.[5] Moreover, this yields an axiomatization of the UES method without Additivity. It appears that Ni and Wang (2007) needed Additivity in Theorem 2 because the Upstream Symmetry property is weaker than Symmetry of the corresponding game solution on $\mathcal{A}^N$. This Game-Symmetry implies the stronger axiom for cost allocation methods that agent i and all its upstream agents pay the same if all agents upstream to i incur no cost. To avoid confusion with Symmetry for game solutions, we call this Cost-Symmetry. Similar as in van den Brink (2004) for auction games, we show that the UES method can be characterized without the Additivity axiom by strengthening the Upstream Symmetry axiom to Cost-Symmetry.

Cost-Symmetry For any $i \in N$ and $C \in \mathbb{R}^n_+$ it holds that $x_j(C) = x_k(C)$ for all $j, k \leq i$, whenever $c_j = 0$ for all $j < i$.

Theorem 2.3 *The UES method is the unique method that satisfies Efficiency, Cost-Symmetry and Independence of Upstream Costs.*

Proof. It is straightforward to verify that the UES method satisfies Cost-Symmetry, the other two properties follow from Theorem 2. To show uniqueness, suppose that method x satisfies the three axioms and let $C \in \mathbb{R}^n_+$. Note that Efficiency and Symmetry imply that

i and all agents with lower demand than i are assumed to have their own demand, and agents with higher demands than i are assumed to have the same demand as i.

[4]The proof is similar as in van den Brink (2004) for auction games. Let $C, C' \in \mathbb{R}^n_+$ and $i \in N$ be such that $c'_h = c_h$ if $h > i$, and consider $j > i$. Take $S \subseteq N \setminus \{j\}$. If $h > i$ for all $h \in S$ then $m^S_j(w^{C'}) = m^S_j(w^C)$ since $c'_h = c_h$ for all $h \in S \cup \{j\}$. Otherwise, if there is an $h \in S$ with $h \leq i$, then $m^S_j(w^{C'}) = m^S_j(w^C) = 0$ since $w^{C'}(S) = \sum_{h=\min S}^n c'_h = \sum_{h=\min S \cup \{j\}}^n c'_h = w^{C'}(S \cup \{j\})$ and similar $w^C(S) = w^C(S \cup \{j\})$. Solution f satisfying strong monotonicity then implies that $x_i(C) = f_i(w^C) = f_i(w^{C'}) = x_i(C')$, and thus method x satisfies Independence on Upstream Costs. Young (1985) already noted that strong monotonicity implies that the payoff of a player in two different games is equal if all its marginal contributions are equal in the two games.

[5]The same remark can be made about a similar independence axiom used in Moulin and Shenker (1992) in characterizing the serial cost sharing mechanism and Strong Monotonicity of solutions for the corresponding auction games.

$x_i(C^0) = 0$ for all $i \in N$, where $C_i^0 = 0$ for all $i \in N$. We prove that $x_i(C)$ is uniquely determined by induction on i.
Take a $C \in \mathbb{R}_+^n$. Let $C^n = (c_1^n, \ldots c_n^n) \in \mathbb{R}_+^n$ be given by $c_i^n = 0$ for all $i < n$, and $c_n^n = c_n$. Cost-Symmetry implies that all $x_i(C^n)$, $i \in N$, are equal. Efficiency then implies that $x_i(C^n) = \frac{c_n}{n}$ for all $i \in N$. In particular, $x_n(C^n) = \frac{c_n}{n}$. With Independence of Upstream Costs it follows that $x_n(C) = x_n(C^n) = \frac{c_n}{n}$ is uniquely determined.
Proceeding by induction, assume that $x_i(C) = \sum_{h=i}^n \frac{c_h}{h}$ is determined for all $i \geq j+1$. Next, let $C^j = (c_1^j, \ldots, c_n^j)$ be given by $c_i^j = c_i$ for all $i \geq j$ and $c_i^j = 0$ for all $i < j$. Independence of Upstream Costs implies that $x_i(C^j) = x_i(C) = \sum_{h=i}^n \frac{c_h}{h}$ for all $i > j$. Cost-Symmetry implies that all $x_i(C^j)$, $i \leq j$, are equal. With Efficiency it then follows that

$$x_j(C^j) = \frac{1}{j}\left(\sum_{i=j}^n c_i - \sum_{i=j+1}^n x_i(C)\right) = \frac{1}{j}\left(\sum_{i=j}^n c_i - \sum_{i=j+1}^n \sum_{h=i}^n \frac{c_h}{h}\right) = \sum_{h=j}^n \frac{c_h}{h},$$

is uniquely determined. With Independence of Upstream Costs it then follows that $x_j(C) = x_j(C^j) = \sum_{h=j}^n \frac{c_h}{h}$ is uniquely determined. □

Although the LRS method also satisfies Independence of Upstream Costs, Ni and Wang (2007) did not use this axiom in characterizing the LRS method. Consequently, they used four axioms to characterize the UES method, but only three to characterize to LRS method. However, here we characterize the LRS method by replacing in the axiomatization of the UES method given in Theorem 2.3, the Cost-Symmetry property by the No Blind Cost property, yielding comparable axiomatizations of these two methods that differ in only one axiom.

Theorem 2.4 *The LRS method is the unique method that satisfies Efficiency, No Blind Cost and Independence of Upstream Costs.*

Proof. It is straightforward to verify that the LRS satisfies Independence of Upstream Costs, the other two properties follow from Theorem 1. To show uniqueness, suppose that method x satisfies the three axioms and let $C \in \mathbb{R}_+^n$. Note that No Blind Cost implies that $x_i(C^0) = 0$ for all $i \in N$, where $C_i^0 = 0$ for all $i \in N$. Similar as in the proof of Theorem 2.3, we prove that $x_i(C)$ is uniquely determined by induction on i (but using No Blind Cost instead of Cost-Symmetry).
Take a $C \in \mathbb{R}_+^n$. Let $C^n = (c_1^n, \ldots c_n^n) \in \mathbb{R}_+^n$ again be given by $c_i^n = 0$ for all $i < n$, and $c_n^n = c_n$. No Blind Cost implies that $x_i(C^n) = 0$ for all $i < n$. Efficiency then implies that $x_n(C^n) = c_n$. With Independence of Upstream Costs it follows that $x_n(C) = x_n(C^n) = c_n$ is uniquely determined.

Proceeding by induction, assume that $x_i(C) = c_i$ is determined for all $i \geq j+1$. Next, let $C^j = (c_1^j, \ldots, c_n^j)$ again be given by $c_i^j = c_i$ for all $i \geq j$ and $c_i^j = 0$ for all $i < j$. Independence of Upstream Costs implies that $x_i(C^j) = x_i(C)$ for all $i > j$. No Blind Cost implies that $x_i(C^j) = 0$ for all $i < j$. With Efficiency it then follows that $x_j(C^j) = \sum_{i=j}^n c_i - \sum_{i=j+1}^n x_i(C)$, and by the induction hypothesis we have $x_j(C^j) = \sum_{i=j}^n c_i - \sum_{i=j+1}^n c_i = c_j$. With Independence of Upstream Costs it then follows that $x_j(C) = x_j(C^j) = c_j$ is uniquely determined. □

We remark that the LRS method also satisfies a reverse of Independence of Upstream Costs, namely *Independence of Downstream Costs* stating that cost shares do not depend on the costs of more downstream agents, i.e. for any $i \in N$ and $C, C' \in \mathbb{R}_+^n$ such that $c_h = c'_h$ for all $h < i$, we have that $x_j(C) = x_j(C')$ for all $j < i$. It can be verified that replacing Independence of Upstream Costs by Independence of Downstream Costs in Theorem 2.4 also characterizes the LRS method. However, note that the UES method does not satisfy Independence of Downstream Costs.

We remark that other axiomatizations of the UES- and LRS methods can be found from the literature, for example by applying the axioms of van den Brink and Gilles (1996) for the more general class of games with a permission structure. In terms of these games it is interesting to remark that the DR polluted river game is the restriction of the LR polluted river on the *permission structure* where the agents are ordered from upstream to downstream.

2.3 Totally positive games

In subsection 2.1 we have shown that the Propositions 1-4 of Ni and Wang (2007) follow from the literature on TU-games. Realising that v^C is an inessential game and w^C is an auction game or dual airport game, the literature provides several other results. First, note that any game in the classes $\mathcal{I}_+^N$ and $\mathcal{A}^N$ are totally positive games, i.e. any game $v \in \mathcal{I}_+^N \cup \mathcal{A}^N$ can be written as a weighted sum $v = \sum_{\emptyset \neq T \subseteq N} \gamma_T u_T$ with $\gamma_T \geq 0$ for all T. Any totally positive game is convex and thus has a nonempty core. From the literature on totally positive and convex games we obtain the following results, see also Brânzei, Fragnelli and Tijs (2002).

Theorem 2.5 *For $C = (c_1, \ldots, c_n) \in \mathbb{R}_+^n$, let $v^C \in \mathcal{I}_+^N$ and $w^C \in \mathcal{A}^N$ be the corresponding LR polluted river game, respectively the DR polluted river game. The following statements are true.*

1. *The Core of v^C (respectively w^C) coincides with the Weber set, being the convex hull of all marginal vectors, of v^C (w^C).*

2. *The Selectope of v^C (respectively w^C) coincides with the Core of v^C (w^C).*

3. *The Bargaining set of v^C (respectively w^c) coincides with the Core of v^C (w^C).*

4. *The Kernel of v^C (respectively w^C) coincides with the pre-kernel of v^C (w^C) and consists of a unique point.*

5. *For v^C and w^C there exist population monotonic allocation schemes.*

Statement 1 follows from Shapley (1971) and Ichiishi (1981) and convexity of the games. Statement 2 follows from Derks, Haller and Peters (2000)and totally positiveness of the games. Statements 3 and 4 follow from Maschler, Peleg and Shapley (1972). Finally, statement 5 follows from Sprumont (1990).

Recalling that the nucleolus is always contained in the Kernel, Theorem 2.5 implies the following corollary.

Corollary 2.6 *The Core, Weber set, Selectope, Bargaining set, Kernel and pre-kernel of v^C all coincide and consist of a unique element, being the nucleolus (Shapley value, τ-value). The Core, Weber set, Selectope and Bargaining set of w^C all coincide. Further, the Kernel and pre-kernel of w^C coincide and consist of a unique element, being the nucleolus.*

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