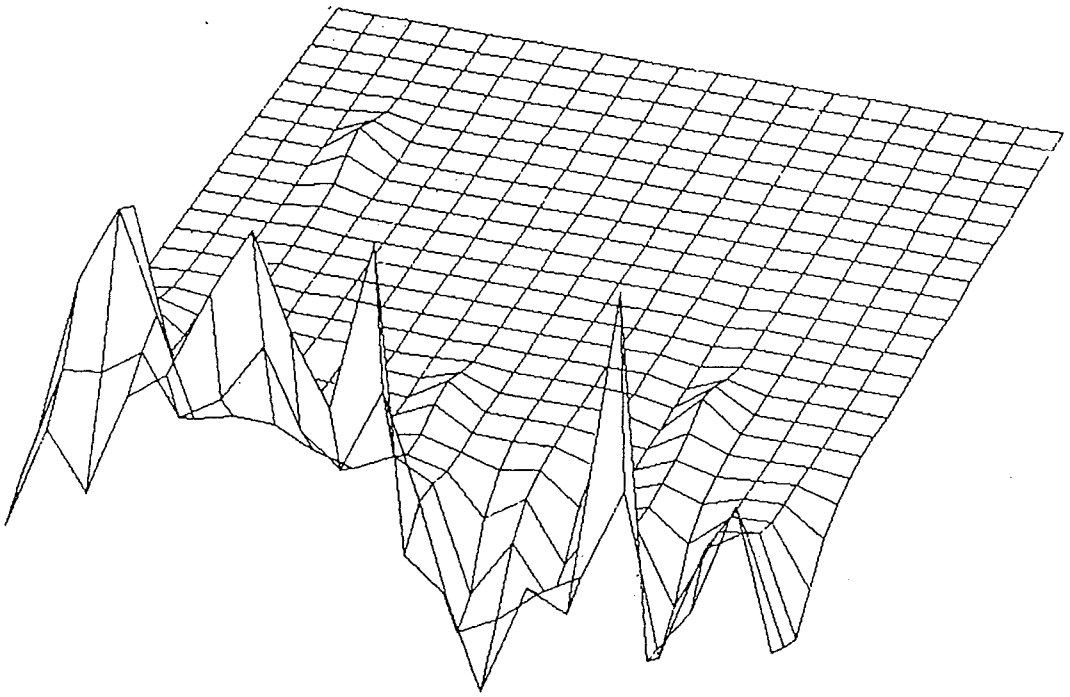
STRETCHED EXPONENTIAL RELAXATION CAUSED BY A DISTRIBUTION OF WAITING TIMES



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august 1987

Abstract

We study the relaxation to equilibrium of a strongly damped charge density wave in the presence of a pinning potential. Numerical simulations on a computer yield a decay following the Williams-Watts-Kohlrausch law $\exp[-(t/\tau)^{\alpha}]$ with $\alpha \simeq \frac{1}{2}$, and lead to an analytical treatment of single points, thereby introducing the concept of waiting times. By introducing a stochastical term to represent the elastic coupling with neighbouring sites, the system can be described by an effectively one particle approximation. By averaging over all possible phases, and therefore over all waiting times, we retrieve the N particle system and are able to theoretically confirm the stretched exponential law.

Acknowledgement

At this place I would like to express my feelings of gratitude towards various persons without whom this work would, at least be of less value, or at worst not exist.

İlk önce, Ayşe Erzan'dan pek çok yardim aldım. Her zamanda beni teşvik etti ve bana yeni fikirler yetişti.

Secondo, per mezzo di Luciano Pietronero efficace aiuto, non mai stancatosi in onorarmi, s'è in ultimo data in luce.

Ten slotte wilde ik alle anderen die op enige wijze hebben geholpen hartelijk daarvoor danken.

Contents

1 Introduction

1.1 Relaxation phenomena

A subject of intense study these days consists of relaxation phenomena. As early as 1847, Kohlrausch proposed a ubiquitious exponential decay law,

$$\psi(t) = \psi_o \exp[-(t/\tau)^{\alpha}], \qquad (0<\alpha\leq 1), \tag{1.1}$$

in order to describe viscoelasticity[1]. In 1971 Williams and Watts rediscovered this law when investigating dielectric relaxation[2,3], and since then, numerous authors have been concerned with the applicability or with the derivation of this law, which is by now generally known as the Williams-Watts-Kohlrausch (WWK) law.

The most striking feature of this law is the wide range of phenomena in which it can be succesfully applied. This might surprise the reader somewhat, for he should be familiar with the so-called Debye law , which is a special case of the WWK law, namely with $\alpha=1$, and which is simply a solution of a first order linear differential equation, as it is often encountered in mechanical friction and damping:

$$\psi(t) = \psi_o \exp[-t/\tau], \tag{1.2}$$

so that

$$\dot{\psi}(t) \equiv \frac{\partial\psi(t)}{\partial t} = \psi_o \frac{-1}{\tau} \exp[-t/\tau] = \frac{-1}{\tau}\psi(t),$$

or equivalently

$$\dot{\psi}(t) = c\,\psi(t). \tag{1.3}$$

On the other hand, eq.(1.1) is a solution of

$$\dot{\psi}(t) = \psi_o \frac{-\alpha}{\tau}\left[\frac{t}{\tau}\right]^{\alpha-1} \exp[-(t/\tau)^{\alpha}] = \frac{-\alpha}{\tau}\left[\frac{t}{\tau}\right]^{\alpha-1}\psi(t),$$

or

$$\dot{\psi}(t) = c(\alpha)\ t^{\alpha-1}\ \psi(t), \qquad (1.4)$$

an equation that looks not as common as eq.(1.3) does.

Among the areas in which use of eq.(1.1) seems justified are charge density waves (CDWs)[2,5], spin glasses[6] and glasses[7].

1.2 Hierarchical models

Although the WWK law describes properly the observed behaviour of this whole range of seemingly unrelated systems, the physical reason why it does so remains unclear.

A nice property of eq.(1.3) is, that it is a linear equation and consequently, doubling the (generalized) force constant implies relaxing twice as fast:

$$\dot{\psi}(t) = 2c\ \psi(t) = \frac{-1}{\tau/2}\ \psi(t)$$

yields

$$\psi(t) = \psi_o\ \exp[-t/(\tau/2)] = \psi_o\ \exp[-2t/\tau].$$

Eq.(1.4) lacks this property since

$$\dot{\psi}(t) = 2c(\alpha)\ t^{\alpha-1}\ \psi(t) = \frac{-\alpha}{2^{1/\alpha}\tau}\left[\frac{t}{2^{1/\alpha}\tau}\right]^{\alpha-1}\ \psi(t),$$

and therefore

$$\psi(t) = \psi_o\ \exp[-\{t/(\tau/\{2^{1/\alpha}\})\}^{\alpha}] = \psi_o\ \exp[-(2^{1/\alpha}t/\tau)^{\alpha}],$$

so that the scaling factor depends in a non-linear way on α.

Generally, a linear response is regarded as a basic mechanism† for relaxation, and attempts will always be made to build a non-linear system out of a linear one. Indeed, for the WWK law, several mechanisms have been proposed, but most of them only make the streched exponential law seem plausible, instead of deriving it from assumptions, supported by physical rather than phenomenological evidence.

† For a better motivation, the reader is referred to the appendix.

The most convenient way to proceed, is to assume weightfunctions[8] $\omega(\tau)$, chosen in such a way as to give the WWK law in the equation

$$\psi(t) = \int d\tau \omega(\tau) \exp[-t/\tau]. \tag{1.5}$$

It is clear that this a purely phenomenological approach, and that it only displaces the physical difficulties from α towards $\omega(\tau)$. The form of the $\omega(\tau)$ therefore is crucial and what is called for is a persuasive mechanism from which it can be obtained.

The hierarchical model, as was first described by Palmer et al.[9], gives us a form of $\omega(\tau)$ which can be explained by physical properties and which gives, inserted in eq.(1.5) the WWK law.

They use a discrete version of eq.(1.5)

$$\left.\begin{aligned} \psi(t) &= \sum_{n=0}^{\infty} \omega(n) \exp[-t/\tau_n] \\ \text{where} \quad & \\ \tau_{n+1} &\equiv 2^{(\mu_n)} \tau_n \\ \text{and} \quad & \\ \omega_n &\equiv \frac{N_n}{N}, \end{aligned}\right\} \tag{1.6}$$

with N_n the number of independent degrees of freedom belonging to a cluster with relaxation rate τ_n.

They postulate a number of different functional forms for μ_n and N_n presumably arising from hierarchical dynamical constraints. They consider

a. $\mu_n = \mu_o$
b. $\mu_n = \mu_o \exp(-\gamma n)$
c. $\mu_n = \mu_o n^{-p}$
d. $N_{n+1} = N_n/\lambda$
e. $N_n = \mu_n/\alpha$
f. $N_{n+1} = \mu_n N_n/\alpha$

Case (cd) is the one that gives a WWK law.

Another interesting choice is the one with

$$\left.\begin{aligned} \tau_n &= n\tau \qquad \left(\text{i.e. } \frac{n+1}{n} = 2^{(\mu_n)}\right) \\ N_n &= \exp[-n^{\beta}]. \end{aligned}\right\} \tag{1.7}$$

With the saddlepoint method it is readily shown that

$$\psi(t) = \psi_o \exp[-(t/\tau)^{\beta/(\beta+1)}],$$

or substituting

$$\beta = \frac{\alpha}{1-\alpha} \tag{1.8}$$

we find the WWK law

$$\psi(t) = \psi_o \exp[-(t/\tau)^{\alpha}].$$

The physical significance of eq.(1.7) is that the number of points N_n that relax after N_{n-1} has relaxed decays exponentially, and that the relaxation time associated with this channel is equal to τ, but since it had to wait $(n-1)\tau$ before being activated, the effective relaxation time equals $n\tau$.

1.3 Aim of this research

The aim of this work is to test a Hamiltonian model, using numerical and analytical methods. We investigate a discrete one dimensional system with overdamped nearest-neighbour elastic interactions, plus a non-linear, random, 'pinning'-potential, giving WWK-like relaxation for intermediate times. (For very long times the relaxation is once more pure exponential, as it is to be expected[9].)

The particular model chosen was originally devised for the description of the dynamics of CDWs[10], and is presented in chapter 2. We will focus not only on the macroscopic behaviour of the polarization, but in particular on the microscopic contributions it consists of. We have found, on the basis of our numerical work, presented in chapter 3, that a lot of insight is to be gained by considering in detail the relaxation mechanism of the individual sites, and this we do in some detail in chapter 4. There, two important concepts are introduced and developed: a sudden rapid increment of the polarization, called an 'event', and the time after which an event manifests itself, the so-called 'waiting-time'. In chapter 5, we try to treat waiting-times in a statistical way, in order to

obtain a distribution of waiting times, leading to a relaxation law in the form of a stretched exponential. Chapter 6 concludes the work and gives a few suggestions for things still to be investigated.

2 The model

2.1 Model Hamiltonian for pinned charge density waves

In this chapter we present the model we are to investigate in later chapters. The model consists of a one dimensional discrete system, the discretization arising from impurity potentials. First we derive the equation of motion. Following the approach of Pietronero and Strässler[10], we consider the following CDW:

$$\rho(x) = e\rho_o[1 + C\cos(q_o x+\phi(x))]. \tag{2.1}$$

Here $\rho(x)$ is the charge density at location x and $\phi(x)$ the phase at x. $e\rho_o$ represents the charge, q_o the Fermi wavevector in one dimension, and C the amplitude of the CDW. For a Peierls system with a constant density of states per site N_{o_j}, this is a constant and equal to[12]

$$C = \frac{N_o \Delta}{a\lambda\rho_o},$$

where 2Δ is the enrgy gap, a the lattice constant, and λ the constant of phonon-electron coupling.

We will consider a pinned system, which means we have impurities, giving rise to pinning potentials. This situation is realized in nature in one dimensional conductors like $NbSe_3$[13]. The pinning potential is given by

$$V_p = V_o \sum_{j=1}^{N} \rho(x)\ \delta(x-x_j). \tag{2.2}$$

Next we assume a damping mechanism acting only at the impurities, that is we write

$$F_d = \frac{-1}{q_o^2}\frac{\rho_m}{\tau}\sum_{j=1}^{N}\frac{\partial\phi}{\partial t}\,\delta(x-x_j), \tag{2.3}$$

where τ determines the strength of the damping, and we allow an external field E to be applied, which shifts the CDW as a whole:

$$U_E = -\frac{e\rho_o}{q_o} E\phi. \tag{2.4}$$

We will work with phase dynamics, that is, while we hold the amplitude of the CDW constant, we consider the modulations of $\phi(x)$, the position of the CDW, along the chain. For a CDW this is the most interesting mode, for it is the only one yielding a non-zero current[11].

The Hamiltonian for the dynamics of the phase $\phi(x)$ can be written as

$$H = \frac{1}{q_o^2} \int dx [\tfrac{1}{2}\rho_m (\frac{\partial \phi}{\partial t})^2 + \tfrac{1}{2}K(\frac{\partial \phi}{\partial x})^2], \qquad (2.5)$$

and here, it has turned out that a picture of overdamped motion seems most appropriate, and henceforth we may safely neglect the kinetic term.

Now, putting in all ingredients, we have

$$-\frac{K}{q_o^2}\frac{\partial^2 \phi}{\partial x^2} + \sum_{j=1}^{N} \delta(x-x_j)\{-\frac{\rho_m}{q_o^2 \tau}\frac{\partial \phi}{\partial t} + e\rho_o CV_o \sum_{j=1}^{N} \sin[q_o x_j + \phi(x)]\} +$$

$$+ \frac{e\rho_o}{q_o} E = 0 \qquad (2.6)$$

as our equation of motion.

If we integrate this equation between neighbouring impurity sites x_j, we obtain the relatively simple equation of motion for the phases $\psi_j \equiv \phi(x_j)/2\pi$:

$$\frac{\partial \psi_j}{\partial t} = B[\psi_{j+1} - 2\psi_j + \psi_{j-1}] - \sin[2\pi(\phi_j + \psi_j)] + E. \qquad (2.7)$$

The first term in the brackets represents an elastic coupling between the phases ψ_j at neighbouring impurity sites, where we have neglected the randomness in the impurity spacings. We have likewise taken the effective charge at each impurity site to be uniform. The random positions of the impurities are included however, in the random phase $\phi_j \equiv q_o x_j/2\pi$ (ϕ_j is not the variable $\phi(x)$ which appears in the equations above), appearing in the non-linear pinning potential. The effective coupling constant B is given in terms of the quantities defined above, as

$$B = \frac{2\pi K n}{e\rho_o CV_o q_o^3},$$

where n is the one dimensional impurity concentration. The electric field has been normalized by $1/E_o$, where $E_o = CV_o n q_o$.

2.2 Existence of a threshold field

When E is sufficiently high, all ψ_j increase, and if all ψ_j are raised by an amount 1, we cannot distinguish this situation from the previous one on the basis of eq.(2.7). Consequently the process will be repeated continuously, the polarization raising higher and higher. The minimum value of E which initiates this process is called the threshold field, E_{th}.

Above threshold, a system will behave in a periodic way; below it there is always at least one stable state. Usually however, there is a large number of metastable states, besides the genuine ground-state, and the system can be found in quasi-equilibrium in any one of these low lying states. Once the system is in a particular state, it will not go to other states, for we assumed T=0, i.e. no thermal noise.

We are primarily interested in relaxation. This word is understood to be given its literal meaning of going to a state of rest. Our discussion will be based entirely on systems with E sufficiently small. Others[10, 12] have determined the values of E_{th}, and these are reproduced in fig.(2.1).

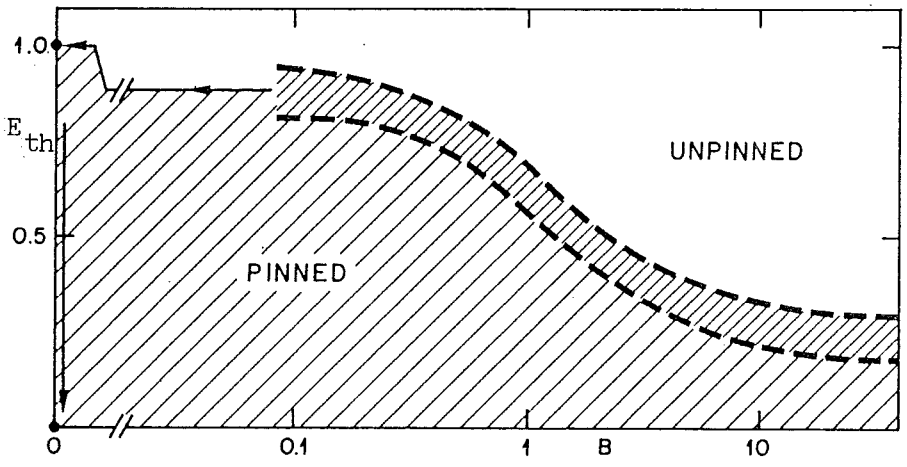


fig.(2.1) The threshold field (reproduced from 12).

2.3 Observable quantities

Since the model was originally devised for CDWs, we can think of each point bearing a unit amount of charge, e. If we take this to be 1, we can define the polarization P by

$$P \equiv \sum_{j=1}^{N} \psi_j \tag{2.8}$$

and with this in mind, we can proceed by defining the current J by

$$J \equiv \frac{\partial P}{\partial t} = \sum_{j=1}^{N} \frac{\partial \psi_j}{\partial t} \equiv \sum_{j=1}^{N} F(\psi_j), \tag{2.9}$$

where we defined a generalized force F by writing the equation of motion (2.7) as

$$\dot{\psi}_j = F(\psi_j). \tag{2.10}$$

Mihály et al.[14] defined a function they called P' as

$$P' \equiv t \frac{\ddot{P}}{\dot{P}} \tag{2.11}$$

because it is independent of P . Since we expect the polarization to follow a WWK law, it is even more suitable to use 1+P', because a plot of this curve versus time offers us a rapid investigation of both α and τ:

$$1+P' = [1 - (t/\tau)^{\alpha}], \tag{2.12}$$

so that the intersection with the 1+P'-axis gives α, while the intersection with the t-axis gives τ.

3 A numerical approach

3.1 Formulation of the problem

Eq.(2.7) is a non-linear equation, and for more than one point (N>1) we obtain N coupled non-linear differential equations, which are clearly too hard to solve by analytical methods. As we will see in chapters 4 and 5, progress can be made by some suitably made approximations, but in order to know what is meant by 'suitably', we first consider the problem numerically. And above all, it is always very instructive to perform simulations on a computer.

There are some difficulties in interpreting the results: clearly a small system can show peculiarities due to non-averaged fluctuations, while on the other hand a large system smoothes everything out.

The next sections describe what the program does and what results it gives. Finally we will discuss how to tackle the analytical problem. For those who want to study the program in detail: it is implemented on the Cyber and written in PASCAL, and it can be obtained by the command 'GET,CDWSIMU/UN=PIETRØ3'.

3.2 Methods

Cyclic boundary conditions (CBC) constitute a generally accepted way to simulate an infinite system by a finite one. In case of long-range interactions the danger of delayed feedback is present, but in our system it turns out that we may safely apply CBC.

The way in which the differential equation will be solved is the usual simplest one, in which the force at each point is evaluated and, accordingly, a tiny time-step dt later, all points are set to their new positions, and the process is repeated. That is to say

$$\frac{\partial \psi}{\partial t} = F(\psi)$$

is written as

$$\lim_{dt \to 0} \frac{\psi(t+dt)-\psi(t)}{dt} = F(\psi),$$

and a small but non-vanishing value for dt is taken. Te smallness of dt is cecked, for instance by comparision with $dt' = \frac{1}{10}\,dt$.

Since $F(\psi_j)$ depends explicitly on j's neighbours, we have to be cautious not to introduce a direction in the algorithm. This is the reason for computing the ψ_j of a whole generation, and only then replacing the old generation by the new one.

We expect some exponential decay, and thus will never reach the equilibrium state, but can approach it as closely as we like by being more patient. From the analytical treatment (see chapter 4), we can extract that it is very unlikely to see anything interesting after $\dot{\psi}$ has decreased beyond, say 10^{-5}. This conclusion we may extend to a many particle system, but it does not apply to the current (which can in principle consist of one large and many small $\dot{\psi}_j$s, and still be considerably below 10^{-5}), but only to the sole $\dot{\psi}_j$ of each j in the sample. Our only criterion for equilibrium is therefore

$$\dot{\psi}_j < J_o, \quad \forall j,$$

and even this J_o is an estimate. When $\dot{\psi}_j$ becomes very small (say $< 10^{-12}\ \psi_j$), the fluctuations in ψ_j are of the order of the last digit, and since this can only change by an integer amount, $\dot{\psi}_j$ at the next step may be disturbed. In conclusion, computational noise might come into play below $\dot{\psi}_j = 10^{-12}$, and we'd better stop the experiment. By then the probability for seeing interesting things is negligible, we hope.

Preparation of the system and ways of averaging

Our system has to be prepared in a certain way, so as to give a relaxation from a well-defined initial state to a final state. For this preparation we use the following method:

a. give all ψ_j a certain value $\psi_j(t=-\infty) = c$,
b. apply a field $E = E_o$,
c. release all ψ_j until $\dot{\psi}_j < J_o$. Now we have our $\psi_j(t=0)$.

For the first and second step there are several possibilities, of course, for the choice of E_o and c. Most experiments were done with $E_o = 0$ and $c = 0$.

In order to get better statistics there are several possibilities (the index (ij) denotes i^{th} sample, j^{th} phase point):

a. take one very long chain,
b. take several chains with $\psi_{ij}(t=-\infty) = 0$ and each point (ij) having a different phase ϕ_{ij},
c. take several chains with $\phi_{ij} = \phi_{0j}$ and $\psi_{ij}(t=-\infty) = \psi_{i0}(t=-\infty)$.

Case b. corresponds to an average over configurations, while case c. is an average over initial states.

Taking one long chain (case a.) turns out to be almost equivalent to dividing the chain in several segments and tying the ends of each (case b.). When using $E_o = 0$, an average over initial states (case c.) also gives the same behaviour. For convenience in notation (avoiding double indices), we always use one chain (case a.).

After preparation comes our experiment:
a. change the field by an amount ε at t=0, i.e. $E(t>0) = E_o+\varepsilon$,
b. observe the quantities as discussed in section (2.3) during the relaxation towards the new equilibrium state,
c. stop the experiment when $\dot{\psi}_j < J_o$, $\forall j$. For the J_o we nearly always choose $J_o = 10^{-10}$.

The field E should be below E_{th}, as discussed in section (2.2), but it should not be too small, so that a clear observation of P(t) and J(t) is possible. For this, we have found that $E \simeq \frac{1}{2}E_{th}$ to be a good choice, and we have studied experiments with $E_o = 0$, $E = \frac{1}{2}E_{th}$, and with $E_o = \frac{1}{2}E_{th}$, $E = 0$.

3.3 Results

For the typical behaviour of pinned CDWs, a set of 500 randomly distributed impurity phases was generated. Unless indicated otherwise, this system is used with a coupling constant B = 1, $\psi_j(t=-\infty) = 0$, $E_o = 0$, and $E = \frac{1}{2}E_{th} = 0.3$. This system is abbreviated as S500. Only a few times, a system is used with 20 or 5 points and with the impurity phases arranged in a special way.

A typical example of the relaxation behaviour, obtained from simulations, is provided by fig.(3.1), which is a representation of the polarization P versus time t of system S500, with superimposed a streched exponential with $\tau = 0.3$ and $\alpha = 0.5$.

A plot of the logarithm of the current J versus time t (fig.(3.2)) is a revealing way to picture some phenomenological properties of the system. First we notice that we can only approximate J by a stretched exponential for intermediate times. Secondly, we observe

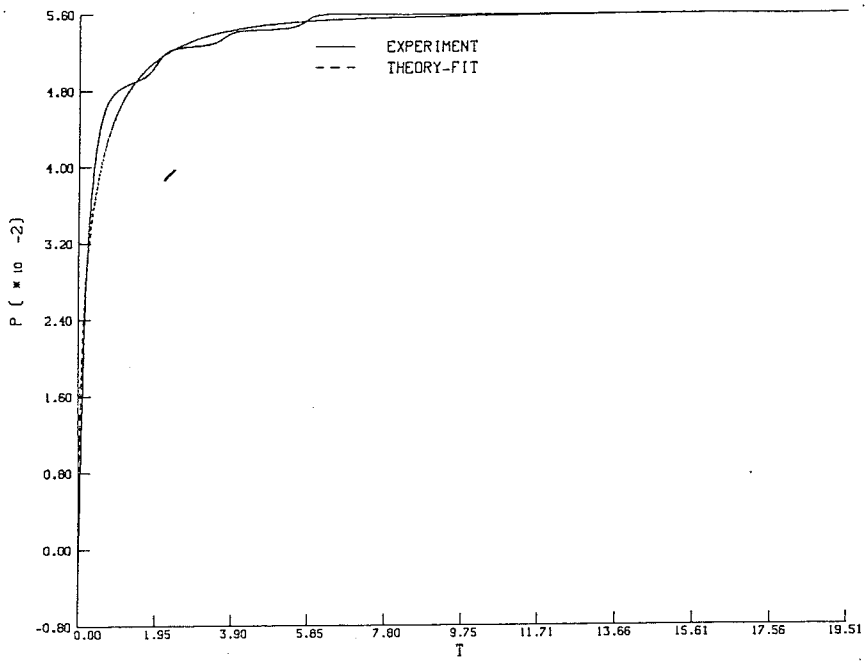


fig.(3.1) Polarization P versus time t, together with a stretched exponential curve $\exp[-(t/\tau)^{\alpha}]$ with $\alpha=0.5$ and $\tau=0.3$.

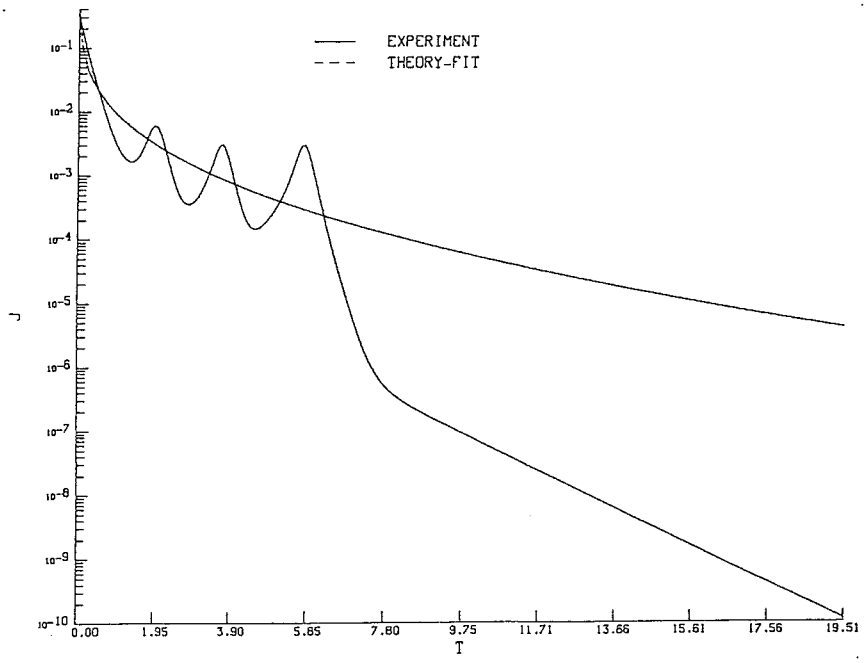


fig.(3.2) Current J versus time t, with the time derivative of the same stretched exponential.

various peaks, more or less pronounced, and occurring at various time intervals. In the tail of the graph there are no more peaks, but an equally sudden change of the slope and therefore of the rate of the decay, appears.

Simulations up to 10000 points have been performed. These last give more peaks covering each other, thereby improving statistics

and fitting in a better way with a stretched exponential with $\alpha \approx 0.5$. At long times, we still see isolated peaks, and finally the transition to the slow relaxation rate, τ_∞.

A straight line in a log(J) versus t plot means Debye relaxation. To be able to read of the relaxation time τ, we define

$$\tau' \equiv \frac{-1}{\frac{\partial}{\partial t}[\ln(J)]},$$

and it is straightforward to prove that in case of Debye decay (eq.(1.2)) we have $\tau=\tau'$. τ' is shown in fig.(3.3). Here we see in a direct manner the equality of the slopes before and after a peak, and the significantly differing τ at the long time tail.

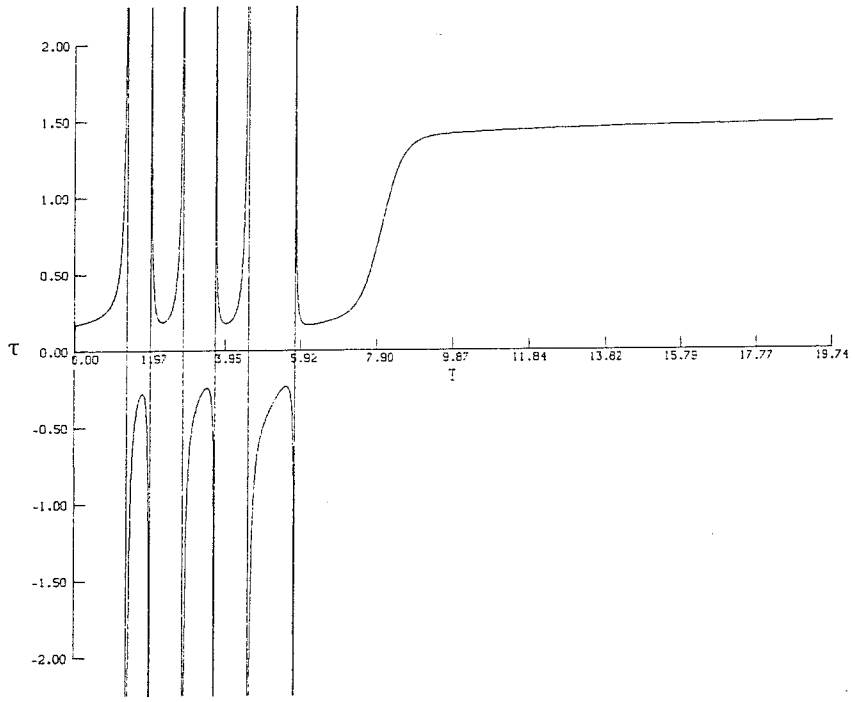


fig.(3.3) τ' = constant means Debye relaxation with $\tau = \tau'$.

The function P' defined by Mihály et al. (eq.(2.12)) turns out to be of little use, since it depends on $\dot{P}$ and $\ddot{P}$ and is therefore very sensitive to small perturbations. Every time a peak occurs, it is disturbed, and at long times the computational noise as mentioned in the previous section becomes evident (fig.(3.4)).

The most intiguing part is the occurence of peaks in fig.(3.2). It looks like the opening of a new channel as was explicitly suggested in the hierarchical model[9]. We will therefore try to understand the origin of the peaks. To study this, we divide the chain into two smaller ones and compaire their behaviour. The two segments are

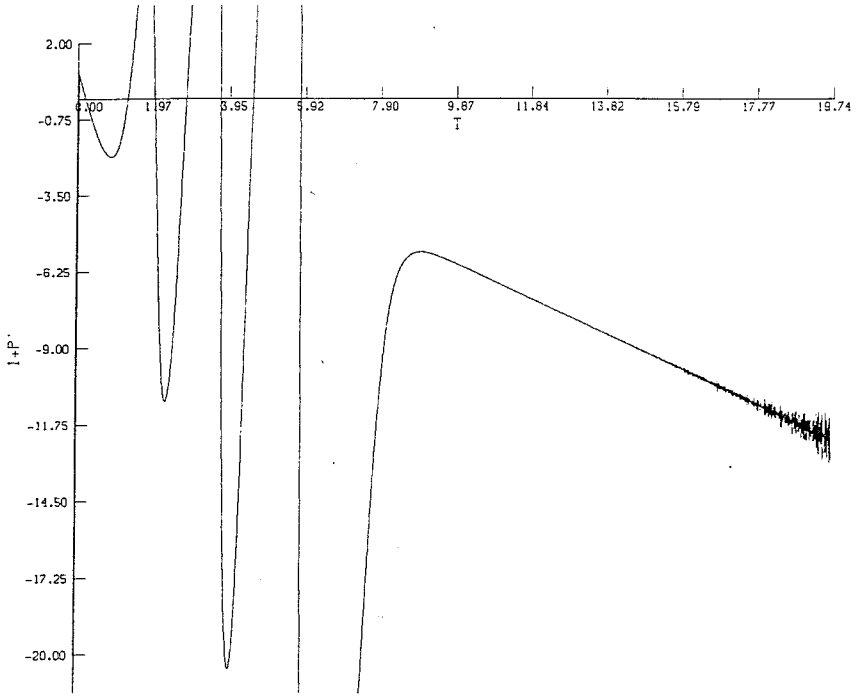


fig.(3.4) P' is of little value.

of equal length (250 points). Fig.(3.5a) shows the current for the first half, fig.(3.5b) for the second half, and fig.(3.5c) for the average of these two. We refer to fig.(3.2) for the two parts tied together.

We notice that fig.(3.5c) is equal to fig.(3.2). This supports our statement in section (3.2) that taking one long chain is equivalent to dividing this chain in segments and tying the ends of each. Furthermore we observe that the peaks and the tail in one half of the cahin contribute almost linearly to the behaviour of the system as a whole. From these observations, we deduce that long-range correlations and long-range hierarchical effects are (nearly) absent.

To proceed in the investigation of the peaks, we make three dimensional plots of $J(j, t) \equiv \dot{\psi}_j(t)$ or its logarithm (j denotes the various points; t the time). These are shown in fig.(3.6a, b) for a specially constructed system, and the current of this system is shown in fig.(3.6c). This special system consists of 20 points (CBC), with the impurity phases arranged in the following way:

j	1	2	3	4	5	6	7	8	9	10
ϕ	0.70	0.60	0.40	0.60	0.70	0.80	0.70	0.30	0.70	0.80
j	11	12	13	14	15	16	17	18	19	20
ϕ	0.20	0.30	0.20	0.60	0.70	0.35	0.70	0.60	0.80	0.75

and $B = 1$, $E_o = 0$, $E = \frac{1}{2}E_{th} = 0.3$.

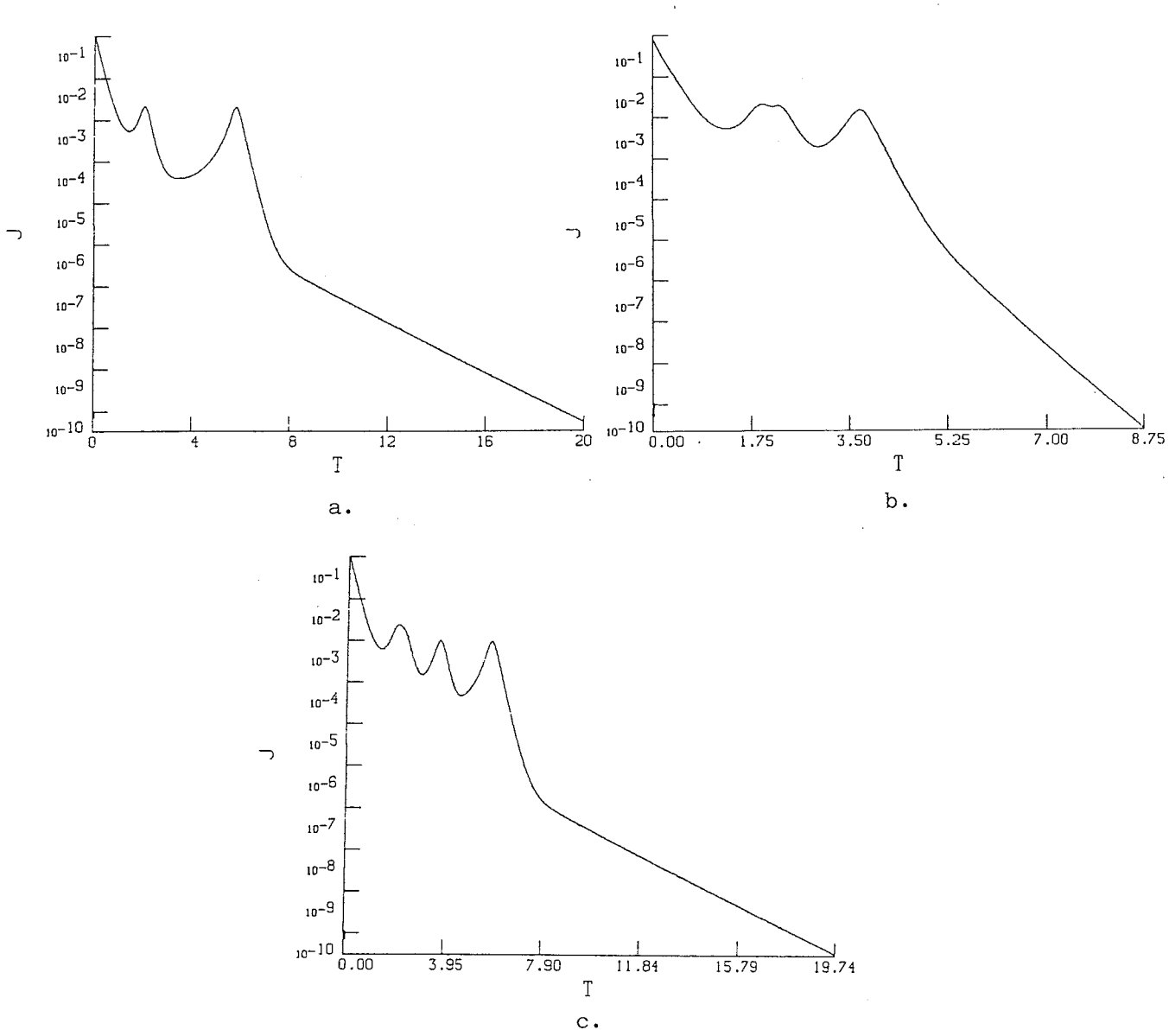


fig.(3.5) Two chains of 250 points (a. and b.) build one larger chain (c.).

Fig.(3.6a) shows the current J for the first 4 time units, and we see that remote parts of the chain suddenly show a peak, and barely influence their neighbours. Fig.(3.6b) shows log(J) for the first 8 time units, and from this, we see that a slow relaxation rate of one part of the chain determines the relaxation rate at long times. At this moment we do not understand why certain combinations of ϕ show peaks or a slow tail.

We knew already from fig.(3.5) that segments of a chain could make important contributions to the current when they either show a peak or have a slow relaxing long time tail. Here we see that both properties are manifestations of individual sites although the reason for this behaviour still can lie in collective effects

(combinations of ϕ seem to be important).

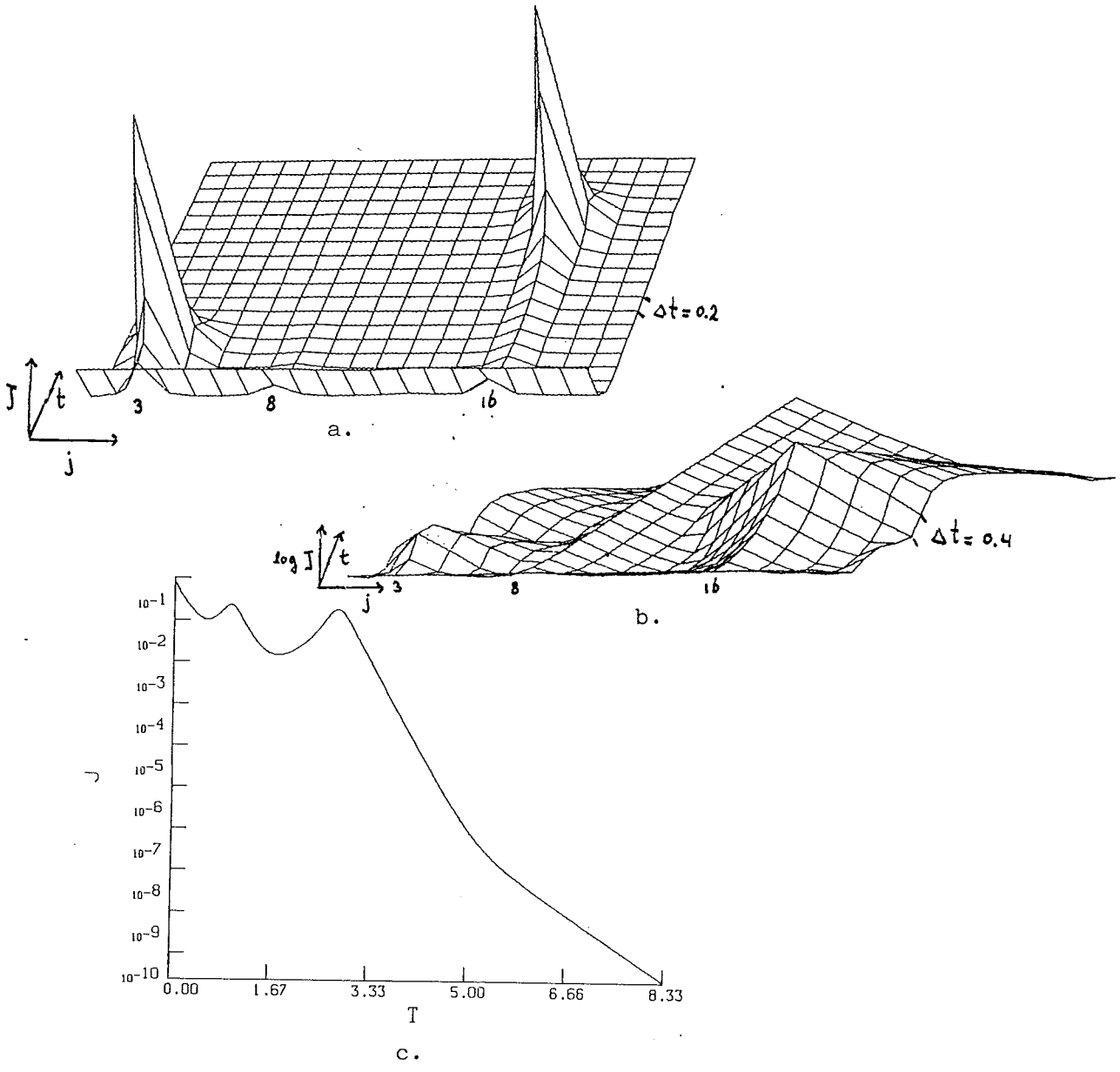


fig.(3.6) Distribution of peaks along the chain (a.), and influence on the long time tail (b.), together with the average of the current (c.).

By taking a very small chain (five or even three points with CBC), we still get a peak if the impurities are properly chosen. Fig.(3.7) illustrates this for a system of 5 points with phases inspired by the 20 particle system:

j	1	2	3	4	5
ϕ	0.60	0.70	0.35	0.70	0.60

with again B = 1, E_o = 0, E = $\frac{1}{2}E_{th}$ = 0.3.

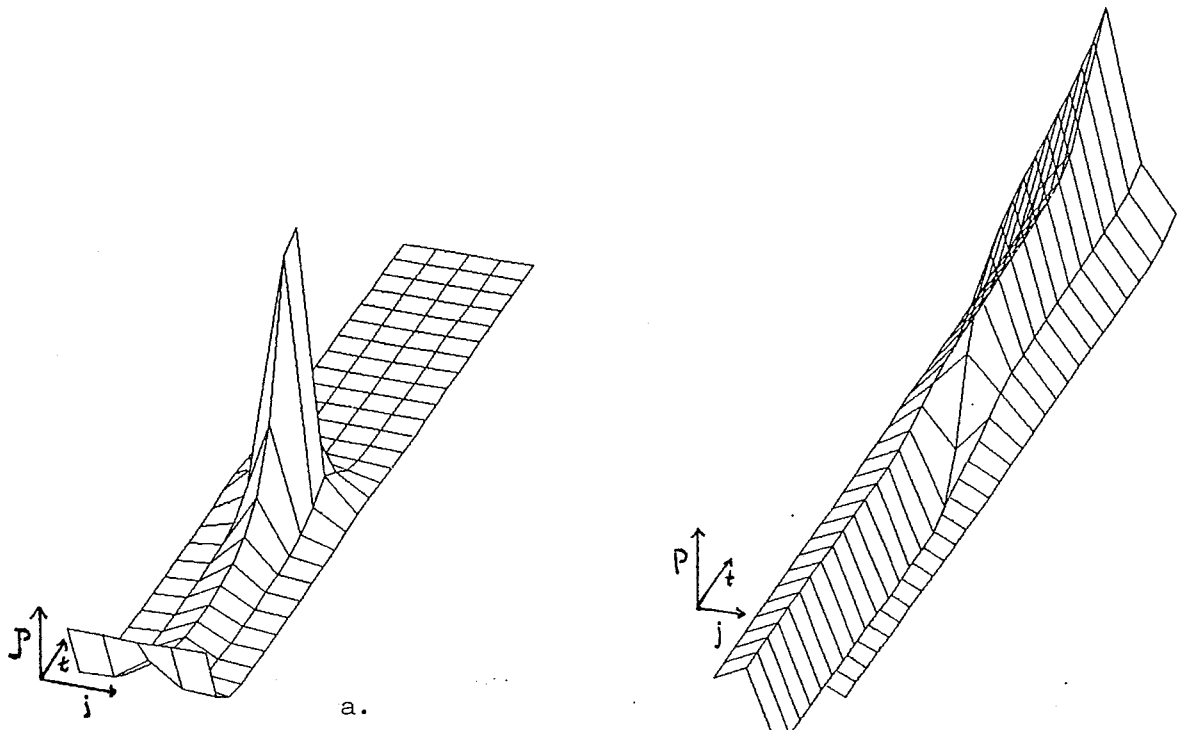


Fig.(3.7) Current (a.) and polarization (b.) in an interesting five particle system.

From this figure we conclude that this peaking behaviour is a very localized thing. And if we look at the polarization P(t), we see that a peak doesn't affect its neighbouring sites too much. A single point seems to move suddenly before its final smooth relaxation towards its equilibrium state. This movement of a point we call an event. Events can be treated semi-analytically by making an effective one particle approximation, and we will do so in section (4.2).

Other values of B

We also looked at other values of B. B = 1 seems to be the most interesting case, since the pinning potential and the neighbour interaction are comparable in magnitude and will compete. For $B \ll 1$ the pinning force will dominate and we expect a rather smooth Debye-like decay. For $B \gg 1$ we expect that there will be stronger short-range correlations and clusters of points on the chain might exhibit more complex relaxational behaviour.

We exhibit the results for B = 0.2 and B = 5 in fig.(3.8). For much smaller values of B the observation of a current becomes problematic (strongly pinned system); for larger values of B the relaxation is very slow, consuming a lot of computation time. Fig. (3.8a) indeed gives a Debye-like relaxation, without peaks.

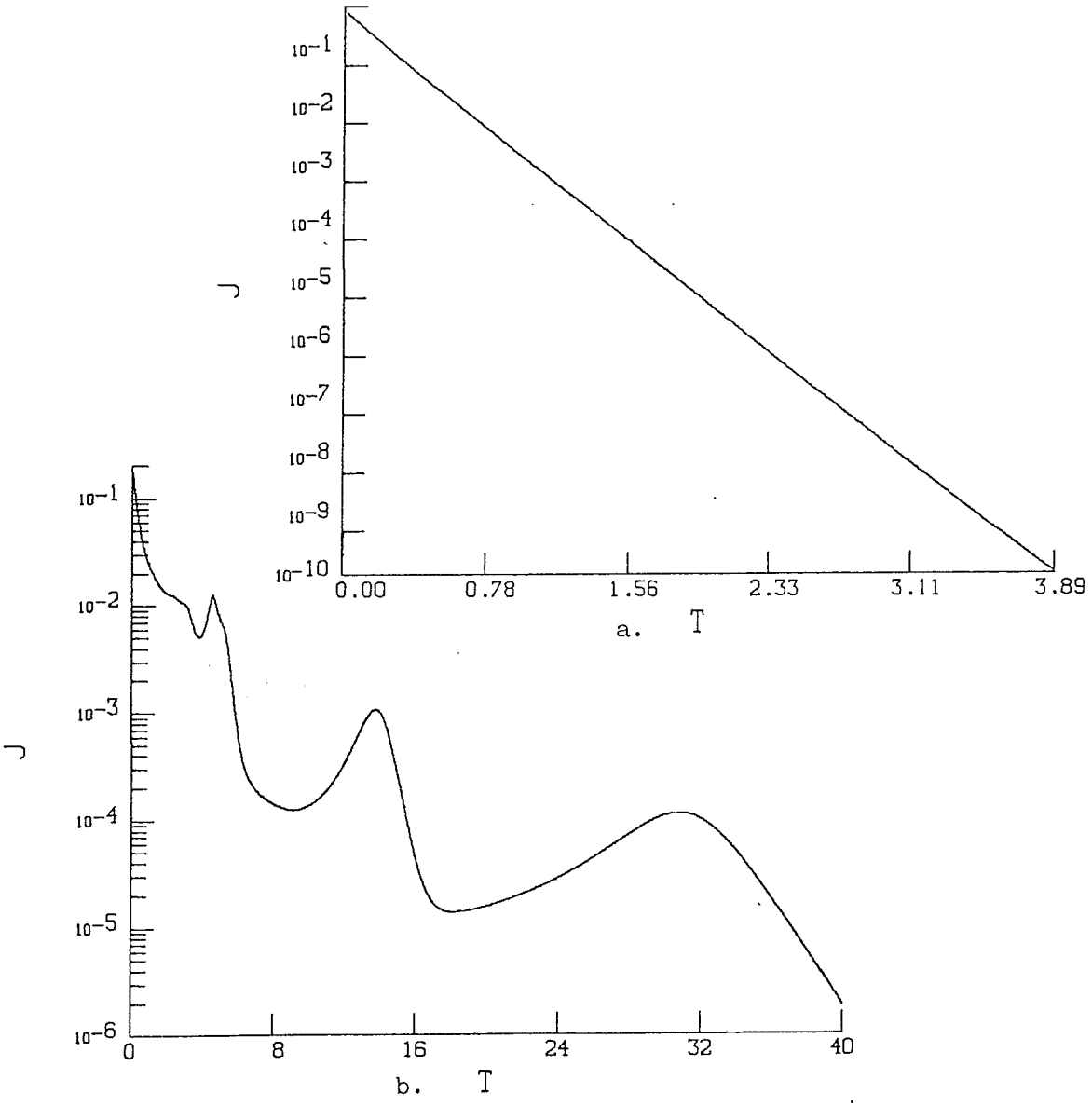


Fig.(3.8) System S500 with (a.) B = 0.2 and (b.) B = 5 shows a different behaviour than with B = 1 (fig.(3.2)).

For large B (fig.(3.8b)) we do observe peaks, but their form does not resemble the peaks in fig.(3.2). We think the relaxation behaviour is far more complicated in this case. It remains to be investigated if this behaviour can be analyzed in terms of a hierarchy of relaxation times.

Samples prepared in non-zero field

We have also investigated what happens if the samples are prepared in the presence of an external field $E_o = \frac{1}{2}E_{th}$, and then at t=0 the field turned off. The subsequent time dependendence of the current is illustrated in fig.(3.9) for the system S500 with B = 1.

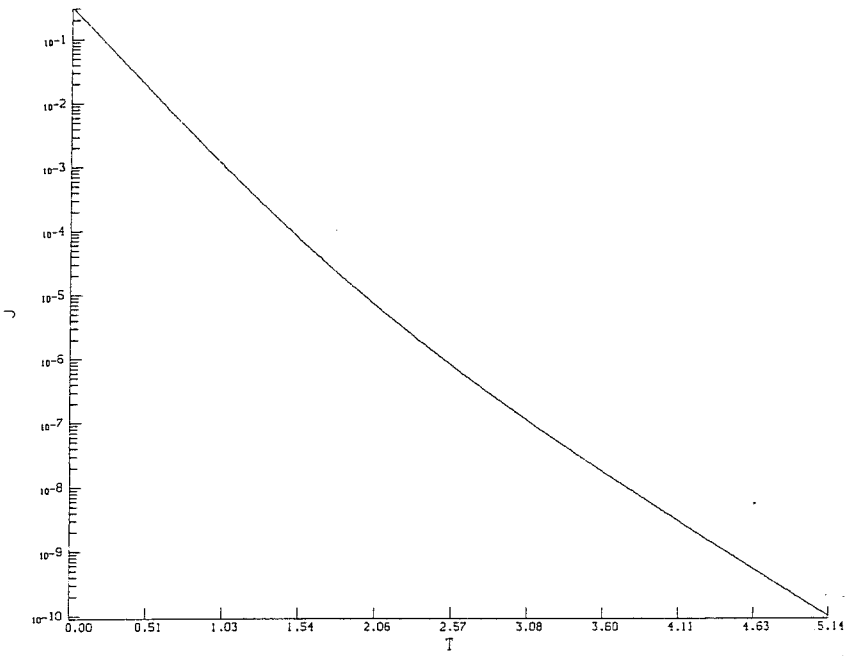


Fig.(3.9) Current when prepared in $E_o = \frac{1}{2}E_{th}$, and $E = 0$.

Observe that the behaviour is not WWK-like but rather Debye-like with a fast and a slow decay rate, with a gradual transition zone in between. In other experiments, the transition was faster, sometimes. The graph resembles the long time part of fig.(3.2).

It is common to observe hysteresis effects in random systems with many stable (and metastable) states, such as the present one. The presence of a field during the initial relaxation seems to drive the system to a ground state, which shifts smoothly with the turning off of the field. This description presupposes a truly collective relaxation phenomenon however, in contradiction to the nearly one particle picture (see chapter 4) in which we have been able to understand the case $E_o = 0$, $E > 0$. Within this one particle approach, it is not possible to explain the difference between the two cases. This point still remains to be clarified.

4 An analytical approach

4.1 Equilibrium states

In this chapter we will use the observation from the numerical work that a single point seems to 'undergo' an event without much influence on its neighbours. Therefore we will study a system consisting of one particle (N=1) with boundaries fixed at 0 ($\psi_{j+1} = \psi_{j-1} = 0$), and drop the index j. The equation of motion (2.7) now reads

$$\dot{\psi} = -2B\psi - \sin[2\pi(\phi+\psi)] + E. \qquad (4.1)$$

It is worthwhile to use a kind of generalized phase space wih base (ψ, $\dot{\psi}$). If we plot eq.(4.1) within this space we get a curve as shown in fig.(4.1).

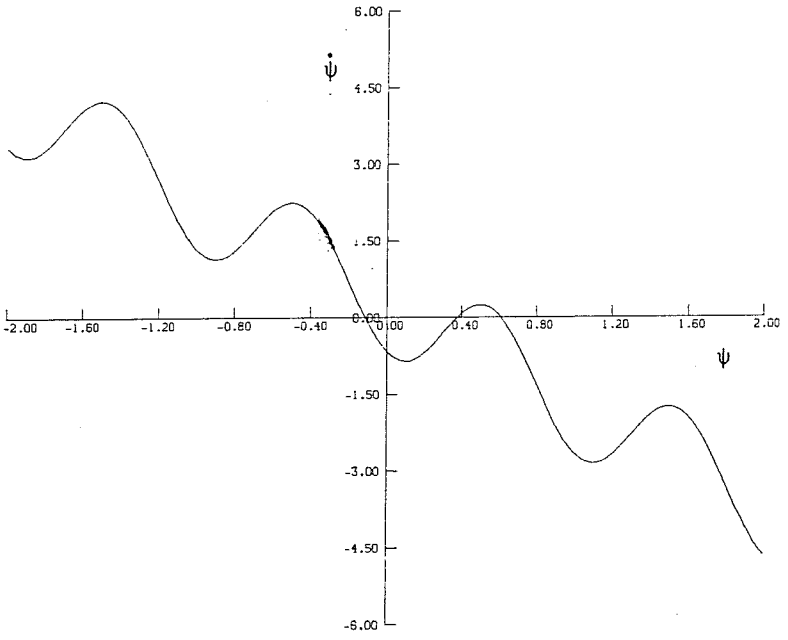


fig.(4.1) Phase plot of eq.(4.1) with B=1, E=0.3, ϕ=0.2.

From this we observe that

a. changing ϕ corresponds to shifting the $\dot{\psi}$-axis,
b. changing E corresponds to shifting the ψ-axis,
c. changing B corresponds to a rotation of the frame,
d. the ψ-axis corresponding to some value of E crosses the function $\dot{\psi}(\psi)$ at least at one point (B>0), and depending on the choice of B, multiple solutions (3(or degenerate 2), 5(or 4), ...) are

possible,

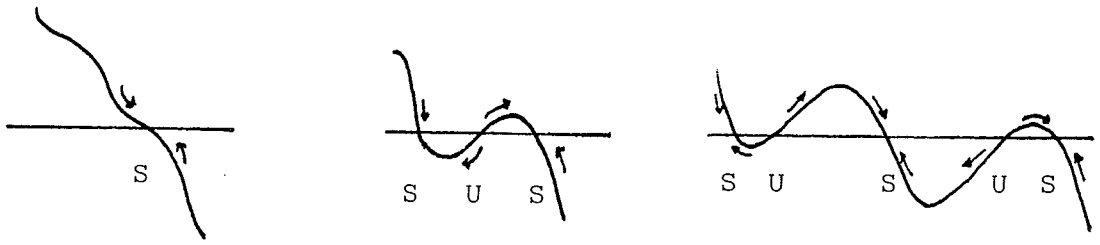
e. the intersections of $\dot{\psi}(\psi)$ with the ψ-axis are fixed points of the system; it is a repulsive fixed point if the slope in a small neighbourhood is positive; otherwise it is attractive. Mathematically:

$$\left.\begin{aligned} &\dot{\psi} = 0,\ \frac{\partial\dot{\psi}}{\partial\psi} > 0 \text{ implies repulsive,} \\ &\dot{\psi} = 0,\ \frac{\partial\dot{\psi}}{\partial\psi} < 0 \text{ implies attractive,} \end{aligned}\right\} \qquad (4.2)$$

f. the initial $\psi(t=-\infty)$ determines at which stable point we will end: if we start between two unstable we will go towards the stable point in between; if we start beyond the last stable one we will go towards the last stable one,

g. there is always at least one stable fixed point ($B\neq 0$).

To gain some intuition it suffices to look at fig.(4.2).



a. b. c.

fig.(4.2) Stable and unstable, single and multiple solutions ($B_a > B_b > B_c$).

It is useful to calculate the location of the extrema of $\dot{\psi}(\psi)$:

$$\frac{\partial\dot{\psi}}{\partial\psi} = -2B - 2\pi\cos[2\pi(\phi+\psi)] \Big|_{\psi=\psi_o} \equiv 0, \qquad (4.3)$$

which implies

$$\cos[2\pi(\phi+\psi_o)] = -\frac{B}{\pi},$$

so that

$$\psi_o = \frac{1}{2\pi}\arccos[-\frac{B}{\pi}] - \phi - m, \qquad m\in\mathbb{Z}, \qquad 0<B\leq\pi.$$

The integer m reflects the fact that the potential is periodic.

To investigate what kind of extrema these ψ_o correspond to, consider

$$\frac{\partial^2 \dot{\psi}}{\partial \psi^2} = 4\pi^2 \sin[2\pi(\phi + \psi_o)] = 4\pi^2 \sin\{\pm\arccos[-\frac{B}{\pi}]\} =$$

$$= 4\pi^2 \sqrt{(1-[\frac{B}{\pi}]^2)} \gtrless 0,$$

and consequently it is a $\{\begin{matrix}\text{minimum}\\ \text{maximum}\end{matrix}\}$. If we now adopt the convention that we write 'maximum' above 'minimum', we have extrema at

$$\psi_o = \mp \frac{1}{2\pi} \arccos[-\frac{B}{\pi}] - \phi - m, \qquad (4.4)$$

which has real solutions only for $B \leq \pi$.

The value of $\dot{\psi}$ at those points is

$$\dot{\psi}_o = \pm\frac{B}{\pi} \arccos[-\frac{B}{\pi}] + 2B(\phi+m) \pm \sqrt{(1-[\frac{B}{\pi}]^2)} + E,$$

and defining

$$C_{\pm}(B) \equiv \pm\frac{B}{\pi}\{\arccos[-\frac{B}{\pi}] + \sqrt{([\frac{\pi}{B}]^2-1)}\}, \qquad (4.5)$$

we write this as

$$\dot{\psi}_o = C_{\pm}(B) + 2B(\phi+m) + E. \qquad (4.6)$$

4.2 Events

Suppose we have a situation as in fig.(4.3a) with $\psi(t=-\infty) > u$.

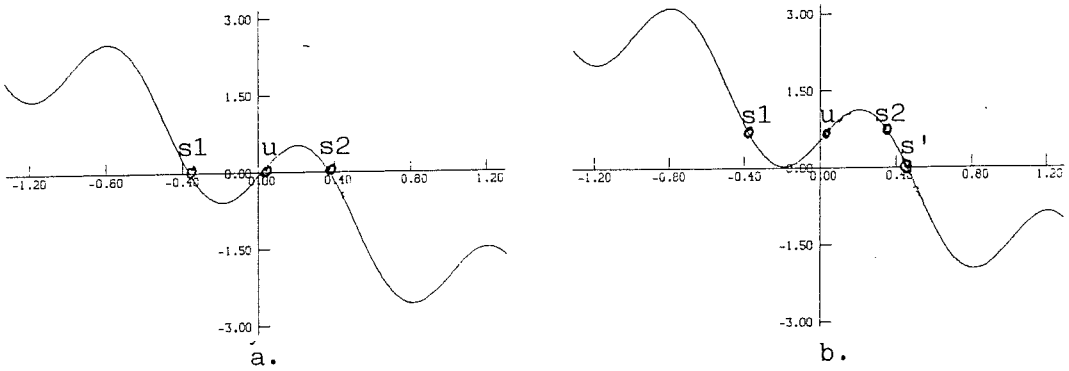


Fig.(4.3) Demonstration of the various scenarios; B = 1, ϕ = 0.49, Ea = 0.0, Eb = 0.6.

The system will relax towards s2. If we now increase the field E by an amount ε such that we get fig.(4.3b), the system will relax smoothly from s2 towards s'.

Next assume $\psi(t=-\infty) < u$. The corresponding equilibrium situation is $\psi(t=0) = s1$, and increasing E now causes $\dot{\psi}$ to decrease at first, but then, before reaching zero current, $\dot{\psi}$ will increase strongly for a while. Afterwards it will again decrease, and when it reaches the point s2, the situation is indistinguishable from the one previously described, and the kind of time dependence of $\dot{\psi}$ is the same.

In fig.(4.4) we show the time dependence of log(J) of the two cases. For this one particle system, P is of course equal to ψ, and J to $\dot{\psi}$. Fig.(4.4b) shows very clear the phenomenon of an event, and now it should be obvious that an event is an observable property of a single point.

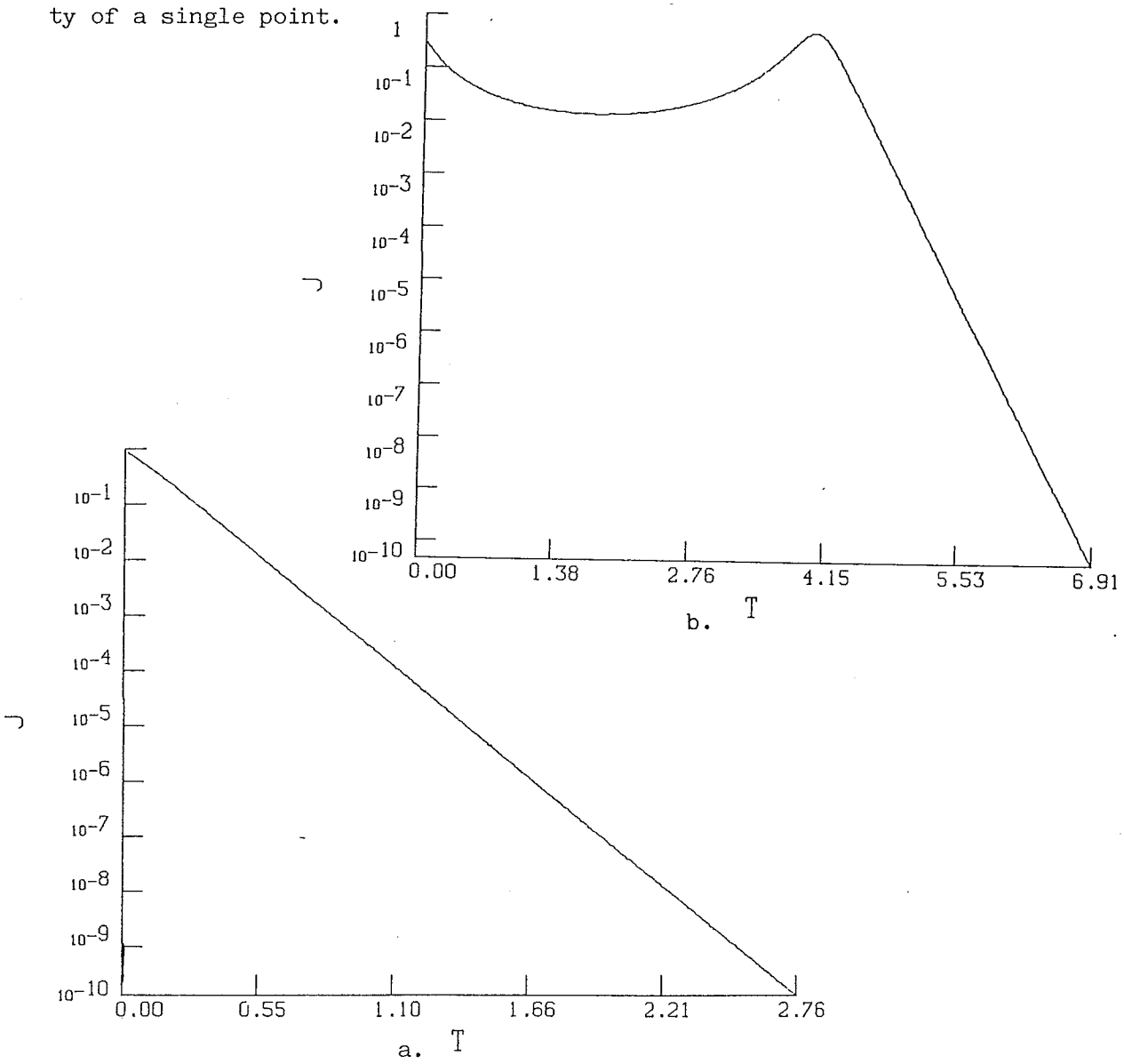


Fig.(4.4) Current of a system (a.) without, and (b.) with an event, corresponding to $\psi(t=-\infty)>u$ and $\psi(t=-\infty)<u$ respectively, obtained by numerically integrating eq.(4.1).

We now can draw some conclusions for the conditions under which an event will take place:

a. for $E = E_o$ (i.e. $t<0$), there must be more than one stable fixed point (i.e. solutions of $\dot{\psi}(s) = 0$ with $\frac{\partial \dot{\psi}}{\partial \psi}\Big|_{\psi=s} < 0$),

b. $E_o \to E_o+\varepsilon$, with $\varepsilon \lesseqgtr \varepsilon_c$, such that a pair of stable (s) and unstable (u) fixed points collide and disappear. Clearly, $\varepsilon_c = \dot{\psi}(\psi_o)$, where ψ_o is the local extremum of $\dot{\psi}(\psi)$ bracketed by u and s,

c. $\psi(t=-\infty) \gtrless u$ if $\varepsilon \lesseqgtr 0$.

In this discussion we left out the case $\psi(t=-\infty) = u$, which would give $\psi(t=0) = u$. The resulting current and polarization then show a third possibility, namely the current possesses a peak without a fall in advance. For a numerically treated system this can in principle occur, but it is not likely to do so. For an anlytical treatment it can, of course. Therefore, we won't analyze it in detail.

We can calculate ε_c using eq.(4.4):

$$\varepsilon_c = C_{\pm} + 2B(\phi+m_o) + E_o, \qquad \varepsilon \lesseqgtr 0, \tag{4.7}$$

where E_o denotes the initial field and m_o the value of the integer m at t=0. This $m_o = \left\{ \begin{matrix} \min m\ , & \varepsilon < 0 \\ \max m\ , & \varepsilon > 0 \end{matrix} \right\}$ to guarantee $\psi(t=-\infty) \gtrless u$.

From eq.(4.4) we see that

$$\{m\} = \{m_o \in \mathbb{Z} \mid \frac{C_+ + E_o}{2B} + \phi < m_o < \frac{C_- + E_o}{2B} - \phi\}, \tag{4.8}$$

and defining

$$\phi'_{\pm}(B, E_o) \equiv \frac{C_{\pm} + E_o}{-2B} \tag{4.9}$$

we have (in case of three solutions of $\dot{\psi}(\psi) = 0$)

$$m_o = \text{int}[\phi'_{\pm} - \phi], \qquad \varepsilon \lesseqgtr 0, \tag{4.10}$$

where int(x) denotes the integer value of x: $\text{int}(\pi)=3$, $\text{int}(-\pi)=-3$.

Inserting eq.(4.10) into eq.(4.7), we obtain after a little algebra

$$\varepsilon_c = 2B[(\phi-\phi'_{\pm}) - \text{int}(\phi-\phi'_{\pm})],$$

or in short

$$\varepsilon_c = 2B[\overset{\text{mod}}{1}(\phi-\phi'_{\pm})], \qquad \varepsilon \gtrless 0. \tag{4.11}$$

From this, the physical significance of ϕ' becomes clear: it is a quantity that depends on the systemparameter B and on the initial field E_o, and selects the values of the phase ϕ (namely those that lie slightly above ϕ') for which an event will be observed upon changing the field by an amount ε.

In conclusion, eq.(4.11) provides a threshold value ε_c (which depends on B, E_o and ϕ) for ε. If our system has an unstable fixed point u for $t<0$, $\psi(t=-\infty) \gtrless u$, and we apply at $t=0$ $E = E_o+\varepsilon$ with $\varepsilon \lessgtr \varepsilon_c$, we will once see an event.

4.3 Rates of relaxation

By expanding the function $\dot{\psi}(\psi)$ near a fixed point $\psi=s$, we can try to make an estimate of τ_∞, the relaxation time as $t\to\infty$. Near $\psi=s$ we can approximate $\dot{\psi}(\psi)$ by a Taylor expansion:

$$\dot{\psi}(\psi) \simeq \dot{\psi}(s) + \frac{\partial\dot{\psi}(\psi)}{\partial\psi}(\psi-s), \qquad |\psi-s| \ll 1. \tag{4.12}$$

Now, since $\dot{\psi}(s) = 0$ and (cf. eq.(4.3))

$$\frac{\partial\dot{\psi}(\psi)}{\partial\psi} = -2B - 2\pi\cos[2\pi(\phi+\psi)],$$

we can write

$$\dot{\psi}(\psi) \simeq -2[B + \pi\cos[2\pi(\phi+s)]](\psi-s),$$

or

$$\dot{\psi}(\psi) \simeq \frac{-1}{\tau_\infty}\psi + S, \tag{4.13}$$

where

$$\tau_\infty \equiv \left(2[B + \pi\cos[2\pi(\phi+s)]]\right)^{-1} \tag{4.14}$$

and

$$S \equiv \frac{1}{\tau_\infty} s.$$

Eq.(4.13) has a solution of the form

$$\psi(t) = s(\exp[-t/\tau_\infty] + 1), \quad |\psi - s| << 1, \tag{4.15}$$

that is, a normal Debye decay characterized by τ_∞. Notice that ther is a lower-bound for τ_∞:

$$\tau_\infty \geq \tau_o \equiv \frac{1}{2(B+\pi)}, \tag{4.16}$$

but no upper-bound:

$$\tau_\infty \leq \frac{1}{2(B-\pi)},$$

which doesn't have a finite value (eq.(4.13) presupposed $\tau_\infty > 0$, and $0 < B \leq \pi$). This means the system cannot approach s faster than τ_o, but can go very slow. This calculation is supported by observations from the numerical work. Eq.(4.16) also implies that B small (<1) barely influences τ_o.

4.4 Waiting times

We have seen in fig.(4.4b) that there is a waiting time $\tilde{\tau}$ associated with the coming of an event. We can understand this by considering $\psi(t+\Delta t)$ as a function of $\psi(t)$, namely $f(\psi)$. We argue that iterating $f(\psi)$ gives us a good approximate picture for the behaviour of $\psi(t)$, namely for $t_n = t + n\Delta t$, the points $\psi(t_{n+1}) \equiv \psi_{n+1} = f(\psi_n)$ will lie on a trajectory which satisfies eq.(4.1). Fig.(4.5) shows $f(\psi)$.

Whenever $f(\psi)$ lies very close to the diagonal (around $\psi = n$), it takes a lot of iterations to go through this region. We call this region a 'neck'. If we start at $\psi < n$, we have to pass n through the neck, spending a lot of time. Having quit this part, we move much faster, and we observe a rise of the current, still later a fall, until we approach the fixed point s. The width of the neck determines the number of iterations and, with it the waiting time for an event.

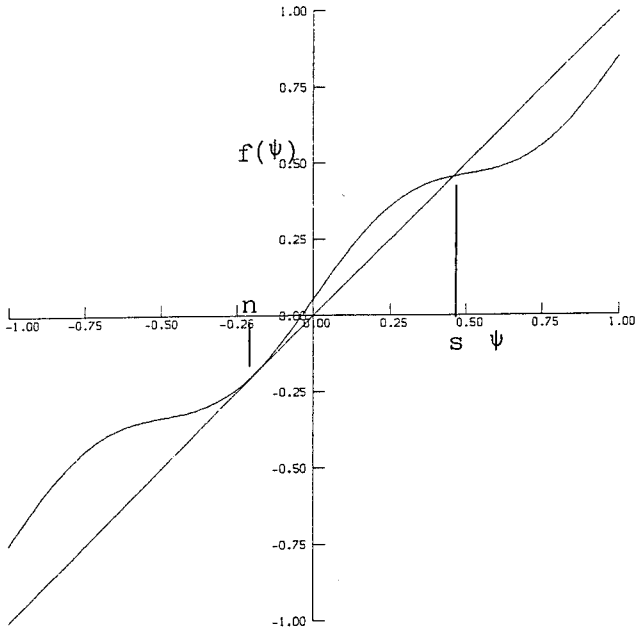


Fig.(4.5) $f(\psi)\equiv\psi(t+\Delta t)$ versus $\psi(t)$; B=1, ϕ=0.49, E=0.6, Δt=0.1.

Until so far, we wrote the equation of motion in the form $\dot{\psi}=F(\psi)$, but we can rewrite this as

$$\psi(t+dt) = \psi(t) + \dot{\psi}(\psi(t))dt$$

or taking dt=1, we have approximately

$$\psi_{n+1} = \psi_n + \dot{\psi}(\psi_n). \tag{4.17}$$

Pomeau and Manneville[15] proved that for a map of the form

$$\psi_{n+1} = \psi_n + \kappa\psi_n^2 + \lambda + O(\psi_n^3) \tag{4.18}$$

around its point of near tangency with the diagonal (here normalized to occur at ψ=0), there exists a laminar region in an interval [-a, a] around the neck, and the time passed within this region is proportional to $|\lambda|^{-1/2}$.

At the point of near tangency ψ_{n° with the diagonal, we have

$$\frac{\partial\psi_{n+1}}{\partial\psi_n} = 1,$$

or using eq.(4.17)

$$1 + \frac{\partial\dot{\psi}}{\partial\psi_n} = 1,$$

and this fulfilled for (cf. eq.(4.4))

$$\psi_{n^\circ} = \mp \frac{1}{2\pi} \arccos[-\frac{B}{\pi}] - \phi - m. \tag{4.19}$$

This means the neck is located at $n=\psi_{n^\circ}$ as given in eq.(4.19).

In the vicinity of $\psi=\psi_{n^\circ}$, we can approximate $f(\psi)$ by a parabola in the following way

$$\psi = \psi_{n^\circ} + \nu, \quad |\nu|<<1. \tag{4.20}$$

This leads to

$$f(\psi) = \pm\frac{B}{\pi} \arccos[-\frac{B}{\pi}] + 2B(\phi+m) - 2B\nu +$$

$$-\sin[\mp\arccos[-\frac{B}{\pi}] + 2\pi\nu] + E\Big|_{\nu=\psi-\psi_{n^\circ}}.$$

Expanding to second order ($\sin(x)\simeq x$, $\cos(x)\simeq 1-\frac{1}{2}x^2$), this yields

$$f(\psi) \simeq (C_{\pm}+2B(\phi+m)+E) + (\mp 2\pi^2\sqrt{(1-[\frac{B}{\pi}]^2)})\nu^2\Big|_{\nu=\psi-\psi_{n^\circ}} \tag{4.21}$$

We see that in eq.(4.18) $\lambda = C_{\pm} + 2B(\phi+m) + E$, and using eq.(4.10), we know the scaling behaviour of λ and hence $\tilde{\tau}$:

$$\tilde{\tau} = 2B[\overset{\text{mod}}{1}(\phi-\phi'_{\pm})]. \tag{4.22}$$

It turns out that the neckwidth λ is exactly equal to ε_c from eq.(4.11), the threshold value for ε to get an event. This should not surprise us.

In conclusion, the time spent in the laminar region, associated with the coming of an event is proportional to

$$\tilde{\tau} \propto |2B[\overset{\text{mod}}{1}(\phi-\phi'_{\pm})]|^{-\frac{1}{2}}, \tag{4.23}$$

depending on whether $f(\psi)$ has a $\{\begin{smallmatrix}\text{maximum}\\ \text{minimum}\end{smallmatrix}\}$ near $\psi=\psi_{n^\circ}$.

5 A statistical approach

5.1 Reduction to a one particle system

Within this chapter we tackle the N particle system. To do so, we first consider, in a way, similar to the one used in chapter 4, N coupled one dimensional maps. Next we use the observation from chapter 3 that for B not too large ($\lesssim 1$), an event hardly influences its neighbours. In this way we can reduce the system of N coupled maps to 1 effectively one dimensional map with a stochastic term. We will treat this map statistically.

We rewrite eq.(2.7) as

$$\psi_{n+1}(j) = \psi_n(j) - 2B\psi_n(j) - \sin[2\pi(\phi(j)+\psi_n(j))] + E + \\ + B[\psi_n(j+1)+\psi_n(j-1)]. \qquad (5.1)$$

This last term can be rewritten in the form

$$B[\psi_n(j+1)+\psi_n(j-1)] = 2B\bar{\psi}_n(j), \qquad (5.2)$$

if we introduce a mean neighbour influence variable

$$\bar{\psi}_n(j) \equiv \frac{\psi_n(j+1)+\psi_n(j-1)}{2}. \qquad (5.3)$$

Versteeg[16] calculated the variance ω^2 for the assumed Gaussian distribution of $\psi_n(j)$ in the absence of an applied field. Its dependence on B is shown in fig.(5.1).

This implies for $\bar{\psi}_n(j)$ a Gaussian distribution with mean value 0 and standard deviation $\sigma=\omega/\sqrt{2}$. In this way we will treat the last term in eq.(5.1) as an independently fluctuating Gaussian variable, thereby reducing the N coupled one dimensional maps to one effectively one dimensional map with additional noise.

The theorem following eq.(4.18) can be extended[17] to a map with noise

$$\psi_{n+1} = \psi_n + \kappa\psi_n^2 + \lambda + O(\psi_n^3) + \sigma\xi_n, \qquad (5.4)$$

where ξ_n is a random term, distributed according to a Gaussian with

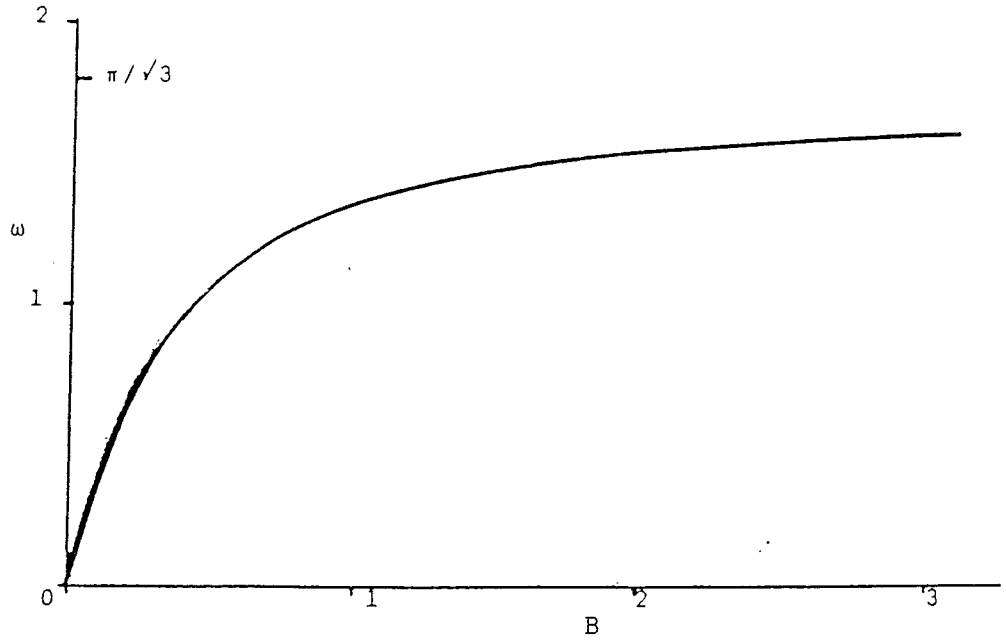


Fig.(5.1) Standard deviation ω of $\psi_n(j)$ (reproduced from 16).

mean value 0 and variance σ^2. Eckmann et al. proved that the time spent in the laminar region is proportional to

$$\tilde{\tau}(\phi_j) \propto T(\kappa, \lambda, \sigma)\ |\lambda|^{-1/2}, \tag{5.5}$$

where the effect of the stochastic term in eq.(5.4) is contained in the coefficient $T(\kappa, \lambda, \sigma)$, depending on the variance of the noise. Notice that $T(\kappa, \lambda, \sigma) \to 1$ as $\sigma \to 0$.

5.2 The distribution of waiting times

Once we are looking at a certain phase point j, ϕ_j is a well-defined constant, although varying randomly from site to site. $\bar{\psi}_n(j)$ fluctuates presumably according to a Gaussian during each iteration of the map for the site j. Therefore, we would like to treat $\tilde{\tau}_j$, the waiting time at this site, in a way similar to the method of section (4.4), and only after having obtained $\tilde{\tau}_j$ integrate over ϕ_j.

Using eq.(5.5) we have

$$\tilde{\tau}_j \equiv \tilde{\tau}(\phi_j) \propto |\varepsilon_{cj}|^{-1/2} \equiv |\varepsilon_c(\phi_j)|^{-1/2}, \tag{5.6}$$

with

$$\varepsilon_c(\phi_j) = 2B[{}^{\mathrm{mod}}_{1}(\phi_j - \phi'_{j\pm})]$$

and

$$\phi'_{j\pm} = \frac{C_{\pm}+E}{-2B},$$

the sign depending on whether $\varepsilon \lesseqgtr 0$.

ε_{cj} can be written in the form

$$\varepsilon_{cj} = a\phi+b.$$

Having thus taken care of the stochastic term in eq.(5.1), we would now like to be able to take an average of the waiting times over the whole chain. For the distribution of waiting times, $P(\tilde{\tau}_j)$, we find

$$P(\tilde{\tau}_j) = \int_0^1 d\phi_j \; P(\phi_j) \; \delta(\phi_j - T\,(a\phi+b)^{-1/2}) =$$

$$= \frac{2T^2}{a}\,\tilde{\tau}^{-3}. \qquad (5.7)$$

5.3 The relaxation law

For an event which sets on at time $t\approx\tilde{\tau}$, we have a rise in the current of the form

$$\exp[(t-\tilde{\tau})/\tau_o],$$

and then a decay characterized by the same relaxation rate, τ_o. Approximating a step-function by

$$1 - \exp[-t/\tilde{\tau}],$$

we have for the average polarization, seen as a linear superposition of such events (cf. chapter 3) with a distribution of waiting times $\tilde{\tau}$,

$$\psi(t) = \exp(-t/\tau_o)\int_0^{\infty} d\tilde{\tau}P(\tilde{\tau})(1-\exp(-t/\tilde{\tau}))\exp((t-\tilde{\tau})/\tau_o),$$

or

$$\psi(t) \propto \text{Const}\int_0^\infty d\tilde{\tau}P(\tilde{\tau})\exp[-(t/\tilde{\tau})-(\tilde{\tau}/\tau_o)] \qquad (5.8)$$

Introducing $x\equiv 1/\tilde{\tau}$, and using eq.(5.7), this is equivalent to

$$\psi(t) \propto \int_0^\infty dx\ \exp[-tx-1/(\tau_o x)] =$$

$$= -\frac{d}{dt}\int_0^\infty dx\ \exp[-tx-1/(\tau_o x)]. \qquad (5.9)$$

This last integral is related to the modified Bessel function K_1 by

$$\int_0^\infty dx\ \exp[-\beta/(4x) - \gamma x] = \sqrt{(\frac{\beta}{\gamma})}\ K_1(\sqrt{(\beta\gamma)}). \qquad (5.10)$$

Here $\gamma=t$ and $\beta=4/\tau_o$.

Eq.(5.9) is now equal to

$$\psi(t) \propto -\frac{d}{dt}\{\frac{2}{\sqrt{(t\tau_o)}}\ K_1(2\sqrt{(t/\tau_o)})\}. \qquad (5.11)$$

Using the expansion for large x

$$K_1(x) \simeq \sqrt{\frac{\pi}{2x}}\ \exp(-x)\ (1+\frac{3}{8x}+\ldots),$$

we write this in the form

$$\psi(t) \propto \frac{d}{dt}\{t^{-3/4}\exp[-2\sqrt{(t/\tau_o)}]\} =$$

$$= (-\frac{3}{4}t^{-7/4} - \frac{2}{\sqrt{\tau_o}}t^{-5/4})\exp[-2\sqrt{(t/\tau_o)}]. \qquad (5.12)$$

In this expression, the exponential factor dominates the behaviour, as is easily seen by considering $\log(d\psi/dt)$, the logarithm of the current, as we considered most of the time in chapter 3:

$$\log(\dot{\psi}(t)) \propto -2\sqrt{(t/\tau_o)} - \frac{7}{4}\log(t), \qquad (5.13)$$

and consequently eq.(5.12) means a stretched exponential law with $\alpha=\frac{1}{2}$, as it was approximately observed in the simulations. There we found for very large times a pure exponential decay, but that is due to the finite size of the system, where there will always be a finite maximum value of $\tilde{\tau}$.

In conclusion, with our assumptions, a Gaussian distribution of $\bar{\psi}_n(j)$ and the form of the relaxation (5.8), we are able to deduce the WWK law $\exp[-(t/\tau)^{\alpha}]$ with exponent $\alpha=\frac{1}{2}$.

6 Conclusions

6.1 Review of the results

Our starting point was a charge density wave on a one dimensional substrate with damping and pinning-potentials. We derived an equation of motion for the phase of the CDW at location x_j, the impurity sites where the CDW is pinned, with an applied field, less than the threshold field E_{th}.

Numerical simulations gave us a stretched exponential with $\alpha \approx \frac{1}{2}$ for the approach to the equilibrium value of the polarization upon application of a field $E<E_{th}$, and a detailed investigation guided us towards a study of the relaxation of single points.

It appeared that for $B \approx 1$ (i.e. elastic coupling of the same order as the pinning-potential), the polarization or current can be approximately seen as a linear superposition of contributions from diffent sites, these contributions consisting of the so-called 'events'.

Just as the simulations were performed by an iteration process, we could approximate the dynamics of the system by a set of N coupled one dimensional non-linear iterated maps, which are representations of N discrete recursion relations.

By regarding the coupling of the maps (the neighbour interaction) as a stochastic variable, we were able to approximate the N coupled one dimensional maps by one single one dimensional map. This allows us to identify two distinct regimes for the approach to equilibrium of the phase ψ_j at individual impurity sites j, namely, a laminar region of duration $\tilde{\tau}_j$, and then a burst with an exponential rise and subsequent Debye-like decay in $\dot{\psi}_j$, characterized by the same relaxation time τ_o, determined only by the derivative at the fixed point. The time $\tilde{\tau}_j$ spent in the laminar region is referred to as the 'waiting time', while the burst corresponds to an event.

The time dependence of the polarization obtained as an average over all sites of events, occurring after the elapsing of waiting times for which we are able to derive a distribution, is proven to be of the WWK form $\psi(t) \propto \exp[-(t/\tau)^{\alpha}]$ with $\alpha = \frac{1}{2}$.

6.2 Outlines for future research

There are several points we have raised which could be pursued further.

The most promising feature we neglected is taking a large neighbour coupling, $B>\pi$, where the effective one particle approximation breakes down. As was shown in fig.(3.8b), more complicated relaxation is exhibited which we have not been able to analyze yet. Since a movement of a phase point does affect its neighbours very much, the ideas of Palmer et al.[9] (see section (1.2)) can become useful.

Another point of interest is preparing the system in non-zero field and then turning the field off. With our treatment this is completely symmetric to the case we have treated in detail, whereas the simulations show a considerable discrepancy (see fig.(3.9)). Why these differ so much is not yet understood in the light of the analytical results, but it could probably arise from the coupling with the system which is neglected in the effective one particle approach.

Appendix

In section (1.2) we dealt with the superiority of the Debey law in favour of the WWK law, and, related with this, the efforts to build a non-linear mechanism out of a linear one.

The German language has a kind of tradition of expressing things in a beautiful though clear manner, and indeed the following quotation is a nice example of this ability, and gives a better motivation for the advantage of linear equations than we did.

'Linearität bedeutet in der physikalischen Interpretation Superponierbarkeit, und daß viele wichtige physikalische Erscheinungen, z.B. die elektromagnetischen Wellen in Vakuum, superponierbar sind, ist eine Naturgegebenheit, ein Glücklicher Zufall vielleicht, aber keine Folge der menschlichen Unzulänglichkeit bei der Behandlung nichtlinearer Gleichungen. Nun, die Wahrheit dürfte wohl weniger zwischen, als vielmehr über diesen vereinfachenden Standpunkten schweben. Die große Bedeutung der linearen Gleichungen in der mathematischen Physik ist weder ein bloßes Naturgeschenk noch eine Schwäche der Physiker, sondern ein Verdienst der Physiker, die nämlich imstande waren, aus der Fülle der Meßbaren Größen aller Art diejenigen herauszufinden, die als Grundgrößen weitreichender Theoriebildung in Frage kommen konnten. Das sind Informationen, die das Chaos der Erscheinungen keineswegs an jeden neugierigen Beobachter gratis Austeilt; von "der Natur" zur Wellenfunktion der Quantenmechanik ist ein weiter Weg.'[18]

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