Measuring asymmetric stochastic cycle components in U.S. macroeconomic time series

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Abstract

To gain insights in the current status of the economy, macroeconomic time series are often decomposed into trend, cycle and irregular components. This can be done by nonparametric band-pass filtering methods in the frequency domain or by model-based decompositions based on autoregressive moving average models or unobserved components time series models. In this paper we consider the latter and extend the model to allow for asymmetric cycles. In theoretical and empirical studies, the asymmetry of cyclical behavior is often discussed and considered for series such as unemployment and gross domestic product (GDP). The number of attempts to model asymmetric cycles is limited and it is regarded as intricate and nonstandard. In this paper we show that a limited modification of the standard cycle component leads to a flexible device for asymmetric cycles. The presence of asymmetry can be tested using classical likelihood based test statistics. The trend-cycle decomposition model is applied to three key U.S. macroeconomic time series. It is found that cyclical asymmetry is a prominent salient feature in the U.S. economy.

Keywords: Asymmetric business cycles; Unobserved Components; Nonlinear state space models; Monte Carlo likelihood; Importance sampling.

JEL classification: C13, C22, E32.

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1   Introduction

Many aggregate economic time series exhibit cyclical fluctuations. Filters derived from a frequency domain representation of the series are often used in cycle analysis. The popular Hodrick and Prescott (1997) filter,[1] despite criticism about its arbitrary nature, remains widely used, alongside refinements and improvements such as the Baxter and King (1999) and Christiano and Fitzgerald (1999) filters. These band-pass filters are usually designed to isolate the fluctuating components in the series with periods between six and thirty-two quarters. Slower moving components are classified as trend, while faster fluctuations comprise the irregular and seasonal parts of the series. The extracted cycles from band-pass filters are visually appealing, but their optimality characteristics typically break down near the end-points of the series. Most applications of these filters are found in historic cycle analysis, although some constructions for forecasting and confidence bounds have been proposed, e.g. Johnson and Gallego (2003).

In traditional linear autoregressive integrated moving average (ARIMA) models, cyclical behavior is usually implied by estimated model parameters rather than explicitly modelled. The theory on estimation, testing, forecasting and building confidence intervals is well established in ARIMA modelling, but a decomposition of the trend and cycle is not as explicit as in the frequency domain. In ARIMA modelling, the trend is usually eliminated by differencing, resulting in models on growth variables. Cyclical variation in the growth can be inferred from the serial correlation structure. From a frequency domain point of view, taking first differences can be regarded as a low-pass filter which does not seperate the cyclical variation from the higher frequency components.

Structural time series or unobserved components (UC) models represent an attractive alternative time domain modelling technique. Trend, cycles and higher frequency components are explicitly modelled by stochastic processes and estimated from the data using Kalman filter and smoothing algorithms. Similar to filters in the frequency domain, decompositions of separate components are immediately visible, while rigorous methods for estimation, testing and forecasting are well developed. The common cycle specification for macro-economic time series in UC models is constructed from stochastic trigonometric functions, as described by Harvey and Jaeger (1993). A generalization of this specification was studied by Harvey and Trimbur (2003). A higher integration order of the cycle

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[1]The filter was widely implemented after its introduction in a working paper in 1980.

was shown to result in a better approximation to the ideal band-pass filter. In our current paper, we extend the basic stochastic trigonometric cycle specification to account for asymmetries in the cycle.

It is widely believed that the cycle in many economic series are asymmetric in the sense that the expansions and contractions do not occur with the same speed. An early widely quoted quantitative study on the asymmetry in economic cycles was published by Neftci (1984). Given a time series y_t, a Markov process I_t is defined with states representing increases and decreases in y_t. Neftci derives likelihood-based asymmetry tests and posterior odds ratios from the transition probabilities of I_t, but the series y_t is not explicitly modelled. In the empirical investigation significant evidence of asymmetry is found in unemployment rate series of the U.S.

Most of the following work on asymmetric cycles concentrate on the U.S. gross national product (GNP) series. Hamilton (1989) analysed the post-war U.S. GNP series in an influential article using a nonlinear parametric model, specifically, an $ARIMA(r, 1, 0)$ model augmented with a latent Markov switching trend process. The paper mainly focuses on filtering the unobserved regime from the data, and presents evidence of the superiority of the specification compared to linear ARIMA and UC models. Although Hamilton presents his model as an extension of Neftci's approach, the issue of asymmetry is hinted at but not addressed explicitly. More recently, Clements and Krolzig (2003) developed rigorous tests for asymmetry in the Markov switching framework, analysing GNP, investment and consumption growth data from the U.S.

Alternative nonlinear autoregressive (AR) models used in modelling cycle asymmetry are variations of of treshhold autoregression (TAR) models. Tiao and Tsay (1994) and Potter (1995) model U.S. output series with self-exciting TAR models, while Beaudry and Koop (1993) and Pesaran and Potter (1997) use constructed indicators as a treshhold and found significant evidence of asymmetry.

State space models with asymmetric cycles have been employed by Kim and Nelson (1999), Luginbuhl and de Vos (1999), Jesus Crespo Cuaresma (2004), either in combination with Markov switching, or using two regimes based on constructed variables or deterministic functions of past observations. Another example of state space modelling with two cycle regimes is given by Harvey (1989, section 6.5), who based the cycle frequency on the sign of first difference of the filtered cycle. Acemoglu and Scott (1997)

constructed a theoretical model to explain asymmetry based on internal intertemporal increasing returns, and also used a state space model to obtain some empirical evidence.

The cycle model in our paper is based on stochastic trigonometric functions, where asymmetry is modelled by specifying the period of the cycle as a function of the steepness. Rather than abruptly switching between two regimes with two distinct cycle periods, the period changes gradually through a continuous range of values. Since our asymmetric cycle specification is a nonlinear State space model, basic linear Kalman filter methods are inadequate for estimation. We base our inference on Monte Carlo likelihood estimation and importance sampling techniques.

The remainder of our paper is organised as follows. In section 2 we define the asymmetric UC cycle model, and discuss some of its properties. Section 3 contains the State space form and elaborates on estimation methods. An empirical investigation on asymmetries in U.S. macro-economic time series is presented in Section 4. Section 5 concludes.

2 Formulation of asymmetric stochastic cycle components

The basic modelling framework employed in this paper is based on the unobserved components (UC) time series model. Following Beveridge and Nelson (1981), Clark (1989) and Harvey (1989), we assume that many macroeconomic time series can be decomposed into a nonstationary trend μ_t, a stationary cycle ψ_t and an irregular component ε_t. The observed y_t is then modelled as

$$y_t = \mu_t + \psi_t + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2), \qquad t = 1, \dots, n. \tag{1}$$

In this section, we focus on the specification of the cyclical component ψ_t.

2.1 Asymmetric deterministic cycles

A deterministic cycle with amplitude a, phase b and frequency λ can be expressed by

$$\psi_t = a \cos(\lambda t - b), \qquad a, b, \lambda, t \in \mathbf{R}, \quad a \neq 0, \ \lambda \neq 0. \tag{2}$$

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The frequency λ is measured in radians and the period of the cycle is given by $2\pi/\lambda$. The cycle ψ_t is symmetric around its local extrema such that

$$\psi_{\tau+s} = \psi_{\tau-s}, \qquad \dot{\psi}_{\tau+s} = \dot{\psi}_{\tau-s} \tag{3}$$

for all $s \in \mathbf{R}$ and all τ for which ψ_τ is a local minimum or maximum, that is $\partial\psi_t/\partial(\lambda t)|_{t=\tau} = 0$. Since

$$\dot{\psi}_t = \partial\psi_t/\partial(\lambda t) = -a\sin(\lambda t - b), \tag{4}$$

it follows that $\lambda\tau = b \pm k\pi$ for $k = 0, 1, 2, \ldots$. We note that the sign of $\dot{\psi}_t$ indicates whether the cycle ψ_t is ascending or descending, while its magnitude determines its steepness.

An asymmetric cycle can be obtained by varying the frequency λ for different values of t. In the simplest case, the cycle can have different frequencies when ψ_t is ascending or descending. More formally,

$$\psi_t = a\cos(\lambda_t t - b), \qquad \lambda_t = \left\{ \begin{array}{ll} \lambda^a, & \dot{\psi}_t > 0 \\ \lambda^d, & \dot{\psi}_t \leq 0 \end{array} \right. . \tag{5}$$

When $\lambda^a \neq \lambda^d$, condition (3) does not hold and we conclude that the resulting cycle is asymmetric but still periodic.

Instead of using two distinct frequencies, we can allow the frequency to depend on a continuous function of $\dot{\psi}_t$, for example,

$$\psi_t = a\cos(\lambda_t t - b), \qquad \lambda_t = \lambda + \gamma\dot{\psi}_t, \tag{6}$$

specifying the frequency as an affine transformation of the cycle steepness. More generally λ_t can be specified as a globally increasing or decreasing function $f(\dot{\psi}_t)$ of the steepness. However, it is unlikely that very specific forms can be inferred from sparse macro-economic data. We will therefore only consider the simple specification (6), which captures the asymmetry phenomenon in one parameter γ. For positive values of γ, the frequency of the cycle is highest when the cycle ascends at its fastest rate, and lowest when it descends at its fastest rate. Figure 1 illustrates the two asymmetric cycle specifications, together with their derivatives. Notice that in the first specification the discontinuities in the two regimes are not clearly visible in the cycle, but obvious in the derivative.

We note that our specification explicitly models asymmetry in the steepness of the cycle. Sichel (1993) introduced asymmetry in the deepness, or amplitude of the cycle, while McQueen and Thorley (1993) distinguished asymmetry in roundness, such that

positive and negative turning points occur with different acuteness. In the stochastic trigonometric specification, deepness and roundness asymmetry may be incorporated by varying the damping factor and the variance of the disturbance. In our current paper, we limit the asymmetry to steepness, following the earlier tradition of studies on asymmetry.

The deterministic cycles ψ_t in (2) and $\dot{\psi}_t$ in (4) can be expressed as sine-cosine waves, that is

$$\psi_t = \alpha \cos(\lambda t) + \beta \sin(\lambda t), \qquad \dot{\psi}_t = \beta \cos(\lambda t) - \alpha \sin(\lambda t), \tag{7}$$

where $\alpha = a \cos b$ and $\beta = a \sin b$. The reverse transformation is $a = \alpha^2 + \beta^2$ and $b = \tan^{-1}(\beta/\alpha)$. The equivalence follows directly from the first of two trigonometric identies

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \qquad \sin(x \pm y) = \cos x \sin y \pm \sin x \cos y, \tag{8}$$

with $x = \lambda_t$, $y = b$. The cycle ψ_t and its partial derivative $\dot{\psi}_t$ can be expressed via a recursion which follows from repeatingly applying the trigonometric identities (8). This recursive expression is given by

$$\begin{pmatrix} \psi_{t+\delta} \\ \dot{\psi}_{t+\delta} \end{pmatrix} = \begin{bmatrix} \cos(\lambda\delta) & \sin(\lambda\delta) \\ -\sin(\lambda\delta) & \cos(\lambda\delta) \end{bmatrix} \begin{pmatrix} \psi_t \\ \dot{\psi}_t \end{pmatrix}, \qquad \delta > 0, \qquad t = 0, \delta, 2\delta, \ldots, \tag{9}$$

with $\psi_0 = \alpha$ and $\dot{\psi}_0 = \beta$. The recursion is linear in ψ_t and $\dot{\psi}_t$. The recursive expression (9) is elegant since ψ_t and $\dot{\psi}_t$ are evaluated simultaneously.

The asymmetric cycle (6) can be expressed recursively by substituting λ_t for λ in (9). Unlike the expression for the symmetric cycle, this recursion is nonlinear in ψ_t and $\dot{\psi}_t$ due to the dependence of λ_t on $\dot{\psi}_t$ and the mutual dependence of ψ_t and $\dot{\psi}_t$ for different values of t.

2.2 Asymmetric stochastic cycles

A stochastic cycle can be based on (9) by including a damping term ϕ and white noise disturbances, see Harvey (1989). Similarly we can obtain an asymmetric stochastic cycle but with λ in (9) replaced by λ_t of (6) to obtain

$$\begin{pmatrix} \psi_{t+\delta} \\ \dot{\psi}_{t+\delta} \end{pmatrix} = \phi \begin{bmatrix} \cos(\delta\lambda_t) & \sin(\delta\lambda_t) \\ -\sin(\delta\lambda_t) & \cos(\delta\lambda_t) \end{bmatrix} \begin{pmatrix} \psi_t \\ \dot{\psi}_t \end{pmatrix} + \begin{pmatrix} \kappa_t \\ \dot{\kappa}_t \end{pmatrix}, \tag{10}$$

$$\lambda_t = \lambda + \gamma \dot{\psi}_t, \qquad t = 0, \delta, 2\delta, \ldots, \tag{11}$$

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where $|\phi| < 1$ is a damping factor, λ_t is the time-varying cycle frequency and the disturbance vectors are Gaussian noise:

$$\begin{pmatrix} \kappa_t \\ \dot{\kappa}_t \end{pmatrix} \sim \text{NID}\left(0, \sigma_\kappa^2 I_2\right), \qquad t = 0, \delta, 2\delta, \ldots \tag{12}$$

The damping term ϕ ensures that the stochastic process ψ_t is stationary. We note that the frequency λ_t is stochastic as a result since $\dot{\psi}_t$ is a stochastic process. In the absence of shocks and with $\phi = 1$, ψ_t and $\dot{\psi}_t$ reduces to the deterministic asymmetric cycle, while a symmetric stochastic cycle is obtained when $\gamma = 0$. In the latter case $\lambda_t = \lambda$, the process ψ_t follows the autoregressive moving average process ARMA(2,1) with the roots of the autoregressive polynomial in the complex range. This property also holds for the process (10) conditional on λ_t. The unconditional process ψ_t follows a nonlinear ARMA(2,1) process with the autoregressive coefficients also depending on an ARMA processes.

The interpretation of $\dot{\psi}_t$ as the partial derivative of ψ_t with respect to λt is not strictly valid for the stochastic process (10). However, it can be taken as a local proxy for the steepness of the cycle ψ_t.

3 Trend-cycle decomposition: estimation and measurement

3.1 Trend-cycle decomposition model

For an observed macroeconomic time series y_t, with $t = 1, \ldots, n$, we consider the model based decomposition given in equation (1). In contrast to ARIMA type models, the series are modelled without differencing. Therefore the trend component μ_t usually requires a nonstationary process.

In our empirical investigation, we employ a smooth trend specification defined by

$$\mu_{t+1} = \mu_t + \beta_t, \tag{13}$$

$$\beta_{t+1} = \beta_t + \zeta_t, \tag{14}$$

$$\zeta_t \sim \text{NID}(0, \sigma_\zeta^2), \qquad t = 1, \ldots, n, \tag{15}$$

where the initial values β_1 and μ_1 are assumed unknown.[2]

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[2]Estimation using the Kalman filter will require a diffuse initialization, see Koopman and Durbin (2003).

The slope of the trend β_t follows a random walk, driven by the disturbance ζ_t. The resulting model for the trend is also called an integrated random walk. To increase smoothness, using higher integration orders is also possible, and can be considered as model representations of Butterworth filters, see Gomez (2001).

The cyclical component ψ_t is modelled as an asymmetric stochastic trigonometric process given by (10). The cycle is driven by the disturbances κ_t, $\dot{\kappa}_t$. Similar formulations of asymmetric cycles may be considered for the generalised cycle components of Harvey and Trimbur (2003).

The irregular term ε_t is taken as Gaussian noise, $\varepsilon_t \sim \mathrm{NID}(0, \sigma_\varepsilon^2)$. In many aggregated macro economic series this term is vanishingly small. We assume that the disturbances of the different components are mutually independent, and independent of the initial values of the trend and cycle processes.

3.2   State space form

Many common linear time series and econometric models are special cases of the linear state space model. It can be formulated through a state transition equation, which describes the evolution of the hidden state vector α_t, and an observation equation, which defines how the state is related to the scalar or vector valued observation y_t:

$$\alpha_{t+1} = T_t \alpha_t + \eta_t, \qquad \eta_t \sim \mathrm{NID}(0, Q_t) \tag{16}$$

$$y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{NID}(0, G_t), \quad t = 1, \ldots, n \tag{17}$$

with initial state vector $\alpha_1 \sim \mathrm{N}(a, P)$. The observations y_t are specified as a linear transformation of a first order vector autoregressive (VAR(1)) process α_t with additional observation noise ε_t. The state space model can be described as a hidden Markov model, although this terminology is mainly associated with processes with a discrete valued state. Both theoretical arguments and experience suggest that the Markovian property holds for many time series by including sufficient elements in the state vector. In particular, all linear autoregressive moving average processes can be cast in state space form. The noise processes ε_t and η_t in the state space equations are mutually and serially uncorrelated Gaussian processes, and independent from the initial value of the state. The deterministic system matrices T_t, Z_t, Q_t, G_t with appropriate dimensions define the structure of the model. In many applications they are time-invariant. In practice, the system matrices

contain unknown parameters that need to be estimated.

The Kalman filter is a recursive algorithm that calculates the optimal (minimum mean square error) estimates of α_t given past observations in a linear Gaussian state space model. In addition, it provides a quick way to construct the likelihood function of the model. The Kalman smoother provides optimal estimates of α_t conditional on the entire set of observations. If the assumption of Gaussianity of the noise processes is dropped, the optimality of the Kalman filter is weakened to optimality in the class of linear predictors, analogous to the Least Squares method in linear regression models. The constructed likelihood is then used to obtain a Quasi Maximum Likelihood estimator. For completeness, we include the filtering and smoothing recursions in the Appendix. More details and proofs can be found in Harvey (1989) or Durbin and Koopman (2001, Part I).

The symmetric trend-cycle decomposition model can be cast in a linear Gaussian state space form, with $\alpha_t = \begin{pmatrix} \mu_t & \beta_t & \psi_t & \dot{\psi}_t \end{pmatrix}'$ and $\eta_t = \begin{pmatrix} 0 & \zeta_t & \kappa_t & \dot{\kappa}_t \end{pmatrix}'$. The system matrices are given by

$$
T_t = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \phi \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix} \end{bmatrix}, \qquad Z_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \tag{18}
$$

$$
Q_t = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \begin{bmatrix} \sigma_\kappa^2 & 0 \\ 0 & \sigma_\kappa^2 \end{bmatrix} \end{bmatrix}, \qquad G_t = \sigma_\varepsilon^2, \tag{19}
$$

where $\mathbf{O}$ represents a conformant zero matrix. Since this is a linear Gaussian state space model, state estimation and likelihood evaluation can be handled by standard Kalman filter methods.

When considering the model with an asymmetric cycle with frequency $\lambda_t = \lambda + \gamma \dot{\psi}_t$ the model becomes nonlinear. Therefore, we need to consider the nonlinear state space model, where (16) is replaced by

$$
\alpha_{t+1} = T(\alpha_t) + \eta_t \tag{20}
$$

with

$$T(\alpha_t) = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \phi \begin{bmatrix} \cos(\lambda_t) & \sin(\lambda_t) \\ -\sin(\lambda_t) & \cos(\lambda_t) \end{bmatrix} \end{bmatrix} \alpha_t, \tag{21}$$

$$\lambda_t = \lambda + \begin{bmatrix} 0 & 0 & 0 & \gamma \end{bmatrix} \alpha_t. \tag{22}$$

3.3   Importance sampling

In linear Gaussian state space models, parameter estimation is in principle straightforward, as the exact likelihood can be quickly calculated using the Kalman filter. For nonlinear state space models, more elaborate methods are required. The most straightforward approach is the Extended Kalman filter, see Jazwinski (1970) or Harvey (1989). The Extended Kalman filter is a basic first order approximation technique, from which an approximating likelihood can be derived. The filter is relatively simple to implement, and works well for small departures from linearity. However, if a better approximation is required, the obvious approach of using higher order approximations involves quite some effort. Moreover, while it has been widely applied for state estimation, the properties of the likelihood approximation are not well understood.

The main alternatives to functional approximation approaches such as the Extended Kalman filter are numerical integration (Kitagawa (1987)) and Monte Carlo methods, like Markov chain Monte Carlo (Gamerman (1998)), rejection sampling (Tanizaki and Mariano (1998)), particle filtering (Gordon, Salmond, and Smith (1993)), most of which require a considerable amount of computation. In this section we describe a basic Maximum Likelihood estimation derived from a Monte Carlo estimate with importance sampling following ideas of Shephard and Pitt (1997) and Durbin and Koopman (1997).

In a state space model with dataset y, state α and parameters collected in the vector θ, the likelihood function is given by $L(\theta) = p_\theta(y) = \int p_\theta(\alpha, y)d\alpha$, where $p_\theta(\cdot)$ is a probablity density function. In nonlinear state space models $p_\theta(y)$ is usually unknown. The joint density $p_\theta(\alpha, y) = p_\theta(y|\alpha)p_\theta(\alpha)$ can be obtained from the model definition, but direct integration is normally infeasible due to the high dimension of α. A well-known solution is Monte Carlo integration, which operates on the principle of formulating the integral as an expectation and estimating it as the sample mean of simulated variables. In particular,

$L(\theta) = \int p_\theta(y|\alpha)p_\theta(\alpha)d\alpha$ could be estimated by drawing $\alpha^{(i)}$ according to the distribution $p_\theta(\alpha)$ and calculating the mean of $p_\theta(y|\alpha^{(i)})$. However, this naive Monte Carlo likelihood does not free us from the dimensionality problem. The majority of the generated $\alpha^{(i)}$'s will diverge very much from the true α and therefore contribute a negligible amount to the likelihood. In practice this means that a prohibitive amount of draws of $\alpha^{(i)}$ is required in order to obtain an accurate estimate. Ideally, $\alpha^{(i)}$ should be simulated conditional on the observations, i.e., from $p_\theta(\alpha|y)$, but for nonlinear models it is not immediately clear how to accomplish this. In linear Gaussian state space models, algorithms to simulate the conditional state, usually referred to as simulation smoothing, have been developed by, amongst others, de Jong and Shephard (1995) and Durbin and Koopman (2002).

Importance sampling in nonlinear state space models can be implemented by using an approximating linear Gaussian state space model. Writing the densities of the approximating model as $g_\theta(\cdot)$, the likelihood is rewritten as

$$L(\theta) = \int p_\theta(\alpha, y)d\alpha \tag{23}$$

$$= \int \frac{p_\theta(\alpha, y)}{g_\theta(\alpha, y)} g_\theta(\alpha, y)d\alpha \tag{24}$$

$$= g_\theta(y) \int \frac{p_\theta(\alpha, y)}{g_\theta(\alpha, y)} g_\theta(\alpha|y)d\alpha. \tag{25}$$

The first factor $g_\theta(y)$ in the last expression is the likelihood of the approximating model; the integral is the expectation of $p_\theta(\alpha, y)/g_\theta(\alpha, y)$ under the distribution of $g_\theta(\alpha|y)$. Hence, the log-likelihood is estimated by

$$\log \hat{L}(\theta) = \log L_g(\theta) + \log \bar{w}, \tag{26}$$

$$\bar{w} = \frac{1}{N} \sum w_i = \frac{1}{N} \sum_{i=1}^{N} \frac{p_\theta(\alpha^{(i)}, y)}{g_\theta(\alpha^{(i)}, y)}, \tag{27}$$

where $L_g(\theta) = g_\theta(y)$ is the likelihood from the approximating model and $\alpha^{(i)}$ are drawn from $g_\theta(\alpha|y)$ using a simulation smoothing algorithm. The ratios of the true model density and the approximating density $w_i = p_\theta(\alpha^{(i)}, y)/g_\theta(\alpha^{(i)}, y)$ are known as importance weights.

3.4   Linear approximating model

The importance sampling procedure as described in the previous section requires a linear Gaussian state space model as an approximation to the nonlinear model. For the

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asymmetric cycle model, we employ a first order linear approximation of the transition equation. For the asymmetric cycle model, the transition equation is partly nonlinear. The function $T(\alpha_t)$ needs to be linearised only with respect to $\dot{\psi}_t$. This implies that only the third and fourth elements of vector $T(\alpha_t)$ are affected, see (21). The third and fourth elements are given by

$$T_3(\alpha_t) = \phi \cos(\lambda_t)\psi_t + \phi \sin(\lambda_t)\dot{\psi}_t, \qquad T_4(\alpha_t) = -\phi \sin(\lambda_t)\psi_t + \phi \cos(\lambda_t)\dot{\psi}_t, \qquad (28)$$

respectively. For some fixed value $\left(\psi_t^* \quad \dot{\psi}_t^*\right)$ of $\left(\psi_t \quad \dot{\psi}_t\right)$, the linearisation around $\left(\psi_t^* \quad \dot{\psi}_t^*\right)$ is given by

$$T_i(\alpha_t) \approx T_i(\alpha_t^*) + \partial T_i(\alpha_t)/\partial \psi_t|_{\alpha_t=\alpha_t^*} \left(\psi_t - \psi_t^*\right) + \partial T_i(\alpha_t)/\partial \dot{\psi}_t\Big|_{\alpha_t=\alpha_t^*} \left(\dot{\psi}_t - \dot{\psi}_t^*\right), \qquad i = 3, 4, \tag{29}$$

where $\alpha_t^* = \left(\mu_t \quad \beta_t \quad \psi_t^* \quad \dot{\psi}_t^*\right)'$ and

$$\frac{\partial \begin{bmatrix} T_3(\alpha_t) \\ T_4(\alpha_t) \end{bmatrix}}{\partial \left(\psi_t \quad \dot{\psi}_t\right)} = R(\alpha_t) = \phi \begin{bmatrix} \cos(\lambda_t) & \sin(\lambda_t) \\ -\sin(\lambda_t) & \cos(\lambda_t) \end{bmatrix} + \begin{bmatrix} 0 & T_4(\alpha_t) \cdot \partial \lambda_t/\partial \dot{\psi}_t \\ 0 & -T_3(\alpha_t) \cdot \partial \lambda_t/\partial \dot{\psi}_t \end{bmatrix}.$$

For the simple affine transformation we have $\partial \lambda_t/\partial \dot{\psi}_t = \gamma$. It follows that

$$\begin{bmatrix} T_3(\alpha_t) \\ T_4(\alpha_t) \end{bmatrix} \approx \begin{bmatrix} T_3(\alpha_t^*) \\ T_4(\alpha_t^*) \end{bmatrix} - R(\alpha_t^*) \begin{pmatrix} \psi_t^* \\ \dot{\psi}_t^* \end{pmatrix} + R(\alpha_t^*) \begin{pmatrix} \psi_t \\ \dot{\psi}_t \end{pmatrix} \tag{30}$$

$$\approx \gamma \begin{bmatrix} -T_4(\alpha_t^*) \\ T_3(\alpha_t^*) \end{bmatrix} \dot{\psi}_t^* + R(\alpha_t^*) \begin{pmatrix} \psi_t \\ \dot{\psi}_t \end{pmatrix}. \tag{31}$$

This linearised approximation of $T(\alpha_t)$ is used in the nonlinear state space model (20) to obtain the linearised state space model

$$\alpha_{t+1} = h_t^* + T_t^* \alpha_t + \eta_t, \qquad y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2), \tag{32}$$

where

$$h_t^* = \begin{bmatrix} 0 \\ 0 \\ -\gamma T_4(\alpha_t^*)\dot{\psi}_t^* \\ \gamma T_3(\alpha_t^*)\dot{\psi}_t^* \end{bmatrix}, \qquad T_t^* = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & R(\alpha_t^*) \end{bmatrix}, \qquad t = 1, \ldots, n, \tag{33}$$

and $Z_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$ as before. Note that unlike the symmetric model, the linear approximation of the asymmetric model has time-varying system matrices.

Starting with a trial value for α_t^*, repeated evaluation of the first order approximation will converge to the conditional mode of the state of the nonlinear model. The converged model is used as the linear approximating model for the importance sampler. The procedure is described, though not implemented in Durbin and Koopman (2001, chapter 11), where examples are given for models with a nonlinear observation equation or with non-Gaussian disturbances. The likelihood based treatment of a nonlinear state equation using importance sampling is as far a we are aware a novelty in econometrics.

3.5   Importance weights

In the asymmetric cycle model, the nonlinearity only occurs in the transition equation, therefore the observation density $g_\theta(y|\alpha)$ is equal to $p_\theta(y|\alpha)$. The importance weights (26) simplify to

$$w_i = \frac{p_\theta(\alpha^{(i)}, y)}{g_\theta(\alpha^{(i)}, y)} = \frac{p_\theta(\alpha^{(i)})}{g_\theta(\alpha^{(i)})} \tag{34}$$

The parts of the densities associated with μ_t and β_t cancel out, since they are identical in the true and approximating model and independent of the cycle. The log-density of the cycle process in the true model, derived from (10), (12) , is given by

$$\log p_\theta(\psi, \dot{\psi}) = \sum_t \log p_\theta(\psi_{t+1}, \dot{\psi}_{t+1}|\psi_t, \dot{\psi}_t) \tag{35}$$

$$= C - \frac{1}{2\sigma_\kappa^2} \sum_t \left(\left(\psi_{t+1} - T_3(\psi_t, \dot{\psi}_t) \right)^2 + \left(\dot{\psi}_{t+1} - T_4(\psi_t, \dot{\psi}_t) \right)^2 \right) \tag{36}$$

while the log-density of the approximating model for the cycle is given by:

$$\log g_\theta(\psi, \dot{\psi}) = C - \frac{1}{2\sigma_\kappa^2} \sum_t \left(\left(\psi_{t+1} - \gamma(\dot{\psi}_t - \dot{\psi}_t^*)T_4(\alpha_t^*) - \psi_t\phi\cos\lambda_t^* - \dot{\psi}_t\phi\sin\lambda_t^* \right)^2 \right. \tag{37}$$

$$\left. + \left(\dot{\psi}_{t+1} + \gamma(\dot{\psi}_t - \dot{\psi}_t^*)T_3(\alpha_t^*) + \psi_t\phi\sin\lambda_t^* - \dot{\psi}_t\phi\cos\lambda_t^* \right)^2 \right)$$

$$\tag{38}$$

The constant term C will cancel in the evaluation of the importance weights.

The simulated likelihood can be optimised using numerical maximisation routines. When using a quasi-Newton method, care must be taken to ensure that the simulated likelihood has a smooth surface. In particular, the same set of random draws for the disturbances must be used when evaluating the likelihood for different values of θ.

4 Empirical evidence from U.S. economic time series

4.1 Data description

The empirical relevance of asymmetric cycles is considered for three key time series from the U.S. economy: unemployment (Un), gross domestic product (GDP) and gross private domestic investment (Inv). The series are obtained from the publicly available database from the Federal Reserve Bank of St. Louis.[3] The unemployment rate in percentage is a monthly series of civilian unemployment compiled by the Bureau of Labor Statistics. The GDP and investment series are the quarterly chain linked series provided by the Bureau of Economic Analysis. The database-codes of unemployment, GDP and investment are UNRATE, GDP and GDPI, respectively. All three series are seasonally adjusted at the source.

We analysed the three series between 1960 and 2004, using 528 observation for the monthly series and 176 observations for the two quarterly series. The data are plotted in the first panels of figures 2, 3 and 4, together with a smoothed trend estimate. The GDP and investment series can be characterised by a strong trend for the long term while cyclical fluctuations from the trend can be observed. The time series of unemployment is most affected by its cyclical behaviour. It should be noted that both quarterly time series are more cyclical in the 1970s and 1980s than in other years. From the end of the 1980s, the amplitude of the cyclical fluctuations is smaller than in the earlier years. These characteristics in macroeconomic time series have been discussed by Stock and Watson (1993). The monthly time series of unemployment does not have a strong trend and is more subject to typical cyclical dynamics.

4.2 Parameter estimation for symmetric decomposition model

The trend-cycle decomposition model (1) with a symmetric cycle is considered first for the quarterly time series GDP and Inv and for the monthly time series Un. This so-called structural time series model is linear and Gaussian and therefore the Kalman filter can be used to compute the likelihood function for a given value of the parameter vector θ. The likelihood function is maximised with respect to θ using numerical optimisation methods. Based on the resulting maximum likelihood estimates, the unobservable trend μ_t and

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[3]`http://research.stlouisfed.org/fred2/`

cycle ψ_t can be estimated for $t = 1, \ldots, n$ (signal extraction) using the Kalman filter and smoother. These calculations are implemented in the `Ox` programming language of Doornik (2001) using the library of state space function `SsfPack` of Koopman, Shephard, and Doornik (1999).

The parameter vector θ contains the unknown log-variances associated with the irregular, trend and cycle components, $\log \sigma_\varepsilon^2$, $\log \sigma_\zeta^2$ and $\log \sigma_\kappa^2$, respectively. The log-variances are estimated so that variances are always positive. The coefficients $0 < \rho < 1$ and $\omega = 2\pi/\lambda > 2$ in the cycle model are also included in the parameter vector but transformed by $\Phi(\rho)$ and $\log(\omega - 2)$ where $\Phi(\cdot)$ is a cumulative density function from, for example, the normal or the logistic distributions. The estimates of θ for the trend plus symmetric cycle model are presented in Table 1. In the cases of GDP and Inv, the irregular variances are estimated as zero while the trend innovation variance estimates are small. Such estimates are typical for macroeconomic time series, with GDP and Inv as examples. These time series have minor irregular changes and are subject to slowly varying (smooth) trends. For the monthly Un series, the irregular does exist while the trend is kept smooth.[4] The cycle properties of the three series are quite similar. The persistence is in all cases estimated to be close to unity. The length of the cycle ω, does differ somewhat. The cycle length for GDP is approximately 6 years while for Inv and Un the length is longer, closer to 9 and 12 years, respectively.

4.3 Parameter estimation for asymmetric decomposition model

The trend plus asymmetric cycle decomposition model is considered next. The parameter vector θ for the previous model is extended with coefficient γ that needs no transformation. The computation of the likelihood function for a given θ is carried out by the methods described in the previous section. A linear Gaussian approximating model is constructed and samples for the unobservable state vector (with trend μ_t and cycle ψ_t) are generated by the simulation smoothing algorithm. From these samples, the Monte Carlo likelihood function can be evaluated and maximised with respect to θ. Table 1 presents the estimation results of the trend-cycle model for the asymmetric cycles specification, next to the results of the symmetric trend-cycle model. Discussions of the empirical results are given

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[4]A unrestricted estimate of σ_ζ for unemployment results in an overly flexible trend, which obscures the cyclical component.

in the next subsection.

The importance sampling techniques employed for the evaluation of the likelihood function can be justified on the basis of the central limit theorem (CLT). Although the consistency property $\widehat{L}(\psi) \to L(\psi)$ as $N \to \infty$ always applies, the CLT is only valid when the second moment of the importance ratio $p_\theta(\alpha, y)/g_\theta(\alpha, y)$ exists, see Geweke (1989). Koopman and Shephard (2004) suggest to carry out test procedures based on the existence of a variance in the importance sampling procedure. The test statistics arise from extreme value theory. For example, importance weights that are larger than a certain threshold value are assumed to come from a generalised Pareto distribution. After calculating a large number of weights, the existence of the second moment can then be formally tested using standard likelihood based tests on the parameters of the Pareto distribution. Under the null hypothesis of a finite second moment, the Wald and Lagrange Multiplier (LM) tests have a standard Normal distribution, while the likelihood ratio (LR) test is distributed as $0.5(\chi_0^2 + \chi_1^2)$. An alternative test based on the largest order statistics of the weights was developed by Monahan (1993). This statistic has a negative value under the null.

In table 2 the test results are reported for the importance weights for the three series. The tests are calculated for weights generated from 100.000 replications of the state. The tests do not indicate a problem for Inv and GDP series. However, for the unemployment series the existence of a second moment is questionable. This is also evident from the plots of the weights shown in figure 5. Model misspecification is usually the main source of unsatisfactory diagnostics for the importance weights, see the discussion below.

4.4 Empirical evidence of asymmetric cycles

First, we note that all three series exhibit asymmetry in the cycle, as is evident from the significant estimate of the γ parameter. The symmetric model is a special case of the asymmetric model, with restriction $\gamma = 0$. The LR, Wald and LM test of the validity of the restriction is given in table 1. All the statistics indicate that there is significant asymmetry at least at the 5% level. The Unemployment series shows a very large increases in the log-likelihood values. The smallest increase in the log-likelihood is 2.4 points, in the GDP series.

Comparing the symmetric and asymmetric specifications, we observe that in general

the cycle disturbances decrease a little, while there is some increase in persistence. For the GDP and Investment series the parameter λ changes little between the two specifications. In the Unemployment series, the extracted cycle is quite different in the asymmetric estimates. It is evident that the increased flexibility of the model leads to a different decomposition.

The estimated asymmetry parameter γ is positive for Unemployment, which implies short upswings and long downturns. For GDP and Investment the parameter is negative, indicating that periods of growth last longer than that of decline. For Unemployment this result is in line with expectations. In particular, our findings agree with believes of classical economists like Mitchell (1927) and Keynes (1936). However, for output and investment there is less consensus in the literature than for unemployment. For example, Falk (1986)'s application of Nefti's non-parametric tests of unemployment series did not produce significant results for the U.S. GNP. Clements and Krolzig (2003)'s parametric tests found evidence of asymmetry in the GNP and investment series with a three-regime model, while in a two regime specification the asymmetry was insignificant.

Table 1 also includes residual diagnostic tests for serial correlation up to twenty lags (Q(20)) and Normality (N). The Q tests for Investment and GDP are generally satisfactory. The asymmetric specification appears to reduce serial correlation, either the symmetric nor the asymmetric specification show no significance at the 10% level. Normality is rejected for both series, although the asymmetric specification for Investment is a considerable improvement on the symmetric specification. The Normality statistics for the Unemployment series are very large, and Normality is clearly not a good assumption. The asymmetric trend-cycle model does give slightly better results for the residual serial correlation, but this statistic remains significant. The model for unemploment appears to be inadequate and requires a more complete specification for the dynamics.

The last panels of the figures 2, 3 and 4 show that the periods are cyclical, and vary between plausible ranges, generally between five and eleven years. It can also be seen that especially for Investment, the variation in the cyclical component is quite small, especially towards the end. This may account for some difficulties in estimating the likelihood using Monte Carlo methods. It is also evident from the plots that the Unemployment series is quite different in character from the other series: there is no clear direction in it's trend, and the period of its cycle is large. The magnitude of the cycle is also considerably larger

than the those of the other series.

5 Conclusion

In this paper we extend the standard stochastic trigonometric cycle component in UC models to account for asymmetries in the period. Replacing the fixed cycle frequency parameter by an affine transformation of the derivative of the cycle results in a model that can capture the degree of asymmetry by one additional parameter. In contrast to common regime switching specifications, the period varies through a continuous range of values.

The trend plus asymmetric model is presented and estimated in a nonlinear state space form. Parameters estimation in nonlinear state space models is not a trivial problem; we use a Monte Carlo likelihood approach, where the likelihood is interpreted as an expectation of ratio of densities and estimated by averaging the densities evaluated in simulated values of the unobserved components. In order to obtain a estimate with a reasonable number of simulations, importance sampling techniques are used.

The empirical application focuses on three U.S. macro economic time series, unemployment, investment and GDP. We find significant evidence of asymmetry in the three series. The unemployment cycle tends to last longer during declines, while the investment and GDP cycles fall faster than they rise.

Appendix

A linear Gaussian state space model is a general modelling framework that encompasses many commonly used econometric models, such as linear regression, ARIMAX and Unobserved Components. The model is defined by a state equation

$$\alpha_{t+1} = T_t \alpha_t + R_t \eta_t, \qquad \eta_t \sim \mathrm{NID}(0, Q_t), \tag{39}$$

and an observation equation

$$y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{NID}(0, G_t) \tag{40}$$

for $t = 1, \ldots, n$. The initial state α_1 is assumed to have a known distribution $\mathrm{N}(a_1, P_1)$ up to some fixed parameters, or having diffuse elements, while the system matrices T_t, Z_t, G_t, Q_t are non-stochastic.

Both the observations y_t and the unobserved state vector α_t are Gaussian processes. The Kalman filter is a recursive algorithm that estimates the mean and variance of α_t conditional on $y_1, \ldots, y_t$. Starting with a_1, P_1, the estimates are updated through

$$v_t = y_t - Z_t a_t, \qquad F_t = Z_t P_t Z_t' + G_t, \qquad K_t = T_t P_t Z_t' F_t^{-1},$$

$$a_{t+1} = T_t a_t + K_t v_t, \qquad P_{t+1} = T_t P_t T_t' + R_t Q_t R_t' - K_t F_t K_t',$$

where $a_{t+1} = \mathrm{E}(\alpha_{t+1}|y_1, \ldots, y_t)$ and $P_{t+1} = \mathrm{Var}(\alpha_{t+1}|y_1, \ldots, y_t)$.

In econometric applications, the state space model usually depends on unknown parameters in the system matrices. The likelihood function of the Gaussian state space model can be quickly evaluated using the Prediction Error Decomposition

$$\log L(\theta) = -\frac{np}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{n} \left(\log |F_t| + v_t' F_t^{-1} v_t \right), \tag{41}$$

where p is the dimension of the vector y_t and θ is the vector of parameters. The quantities v_t and F_t follow from the Kalman filter recursion.

The Kalman filter predicts α_t conditional on past observations, that is, $y_1, .., y_{t-1}$. Given the filter output, the Kalman smoother estimates α_t conditional on the full sample $y = (y_1, \ldots, y_n)'$ using the backward recursion:

$$L_t = T_t - K_t Z_t, \quad r_{t-1} = Z_t' F_t^{-1} v_t + L_t' r_t, \quad N_{t-1} = Z_t' F_t^{-1} Z_t + L_t' N_t L_t,$$

$$\hat{\alpha}_t = a_t + P_t r_{t-1} \quad V_t = P_t - P_t N_{t-1} P_t,$$

and starting with $r_n = 0, N_n = 0$. The smoothed state $\hat{\alpha}_t = \mathrm{E}(\alpha_t|y)$ provides the minimum mean square error estimates of the latent state, with variance $V_t = \mathrm{Var}(\alpha_t|y)$. For detailed discussions of the state space methodology we refer to Anderson and Moore (1979) and Durbin and Koopman (2001).

State simulation smoothing algorithms generate draws of the state $\alpha = (\alpha_1, \ldots, \alpha_n)'$, conditional on observed data y. A simple algorithm developed by Durbin and Koopman (2002) proceeds by generating unconditional draws α^+ of the state and the associated observations y^+ according to the model (39), (40). The Kalman smoother is applied to both the observed y and the generated series y^+, yielding $\hat{\alpha}$ and $\hat{\alpha}^+$ respectively. The series $\tilde{\alpha} = \hat{\alpha} + \hat{\alpha}^+ - \alpha^+$ are realisations of the conditional distribution $\alpha|y$.

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Table 1: Trend cycle decomposition model estimation results.

Maximum likelihood estimates are reported for the trend plus symmetric cycle and trend plus asymmetric cycle (as) model for U.S. unemployment, inventment and GDP. Square brackets contain 95% confidence intervals. Jarque-Bera Normality (N) and Box-Ljung (Q(20)) serial correlation test are also reported, together with log-likelihood values. The likelihood based Wald, LM, LR are asymptotically χ_1^2 distributed.

	Un	Un (as)	Inv	Inv (as)	GDP	GDP (as)
σ_ε^2	7.70×10^{-4}	1.67×10^{-3}	$-$	$-$	$-$	$-$
σ_ζ^2	1.13×10^{-7}	1.13×10^{-7}	1.23×10^{-5}	1.21×10^{-6}	8.29×10^{-8}	7.91×10^{-8}
σ_κ^2	2.77×10^{-2}	2.48×10^{-2}	2.53×10^{-4}	2.44×10^{-4}	5.60×10^{-5}	5.45×10^{-5}
ϕ	0.988 [0.977; 0,993]	0.989 [0.979; 0.994]	0.963 [0.904; 0.986]	0.968 [0.900; 0.990]	0.950 [0.898; 0.976]	0.953 [0.901; 0.978]
ω	124.9 [96.0; 161.2]	102.9 [82.4; 127.8]	24.0 [19.2; 29.9]	24.0 [19.3; 29.9]	36.2 [26.1; 49.9]	34.8 [25.8; 46.7]
γ	$-$	0.00738 [0.00448; 0.0103]	$-$	-0.36 [-0.64; -0.079]	$-$	-0.91 [-1.70; -0.12]
N	146.1	164.6	12.0	8.0	7.8	7.9
Q(20)	73.5	69.4	28.3	23.8	24.5	23.4
Log-Lik	153.8	163.6	428.5	432.3	584.5	586.9
Wald		24.9		7.6		5.0
LM		26.2		6.3		5.3
LR		19.6		8.6		4.8

Table 2: Testing the existence of the second moment of importance weights.

Based on the largest of 100.000 generated importance weights, a Pareto distribution is fitted by maximum likelihood. Under the null of finite variance, the asymptotic distributions of the Wald and LM statistics are standard Normal, LR is $0.5(\chi_0^2 + \chi_1^2)$, and the Monahan (M) statistic is negative, and the Pareto parameter $\hat{\nu}$ is less than 0.5. The sample variance of the weights is reported as WgtVar.

	M	Wald	LM	LR	WgtVar	$\hat{\nu}$
Un	0.174	2.177	1.299	6.432	159.7	0.619
Inv	-0.060	-1.691	-0.867	3.803	8.487	0.407
GDP	-0.048	-0.715	-0.418	0.752	14.65	0.461

Figure 1: Stylised asymmetric business cycles with derivatives.

The first plot shows a regime switching cycle, based on one frequency during ascend and one during descend. The second plot shows a smooth frequency evolution, where the frequency is an affine transformation of the cycle slope. The solid line depicts the cycle while the dashed line represents a proxy of its steepness.
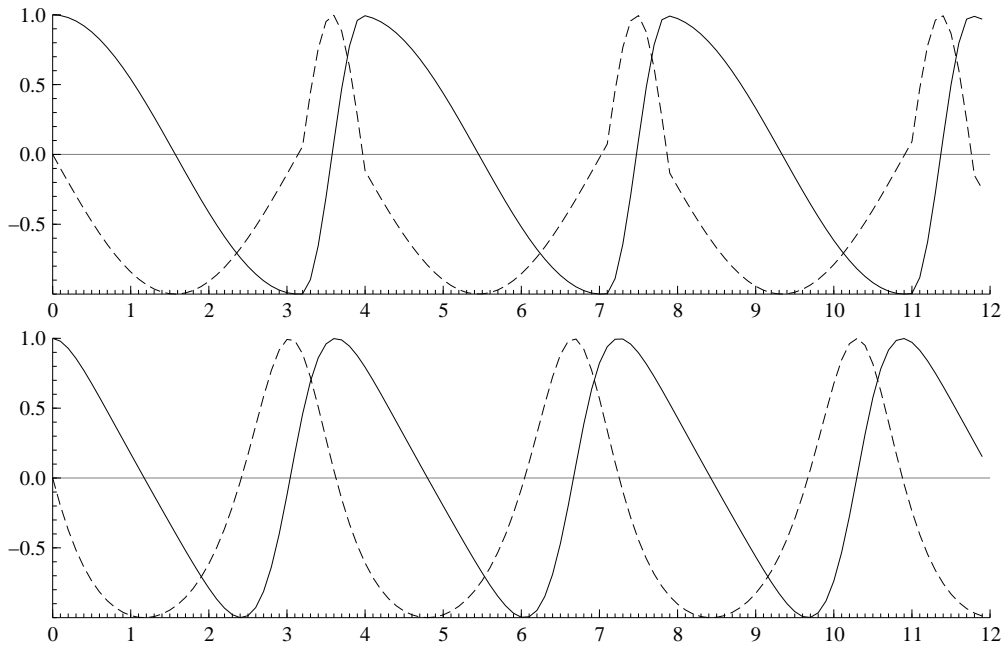

Figure 2: Trend-cycle decomposition of Unemployment.

The first plot shows the data and smoothed trend, the second plot shows the smoothed asymmetric cycle component, the third plot shows the cycle period.
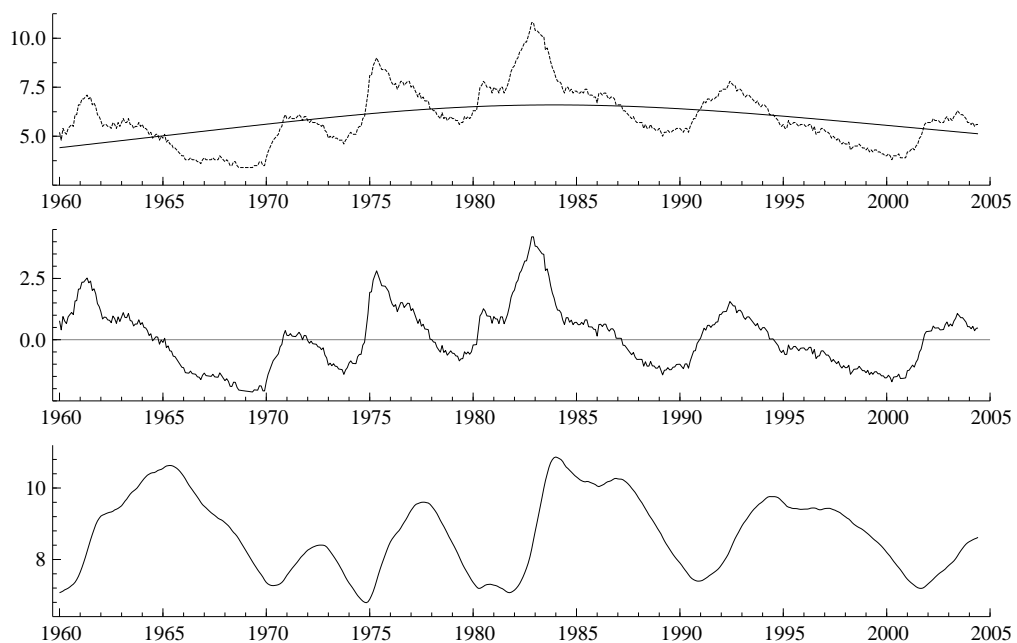
Figure 3: Trend-cycle decomposition of Investment.

The first plot shows the data and smoothed trend, the second plot shows the smoothed asymmetric cycle component, the third plot shows the cycle period.
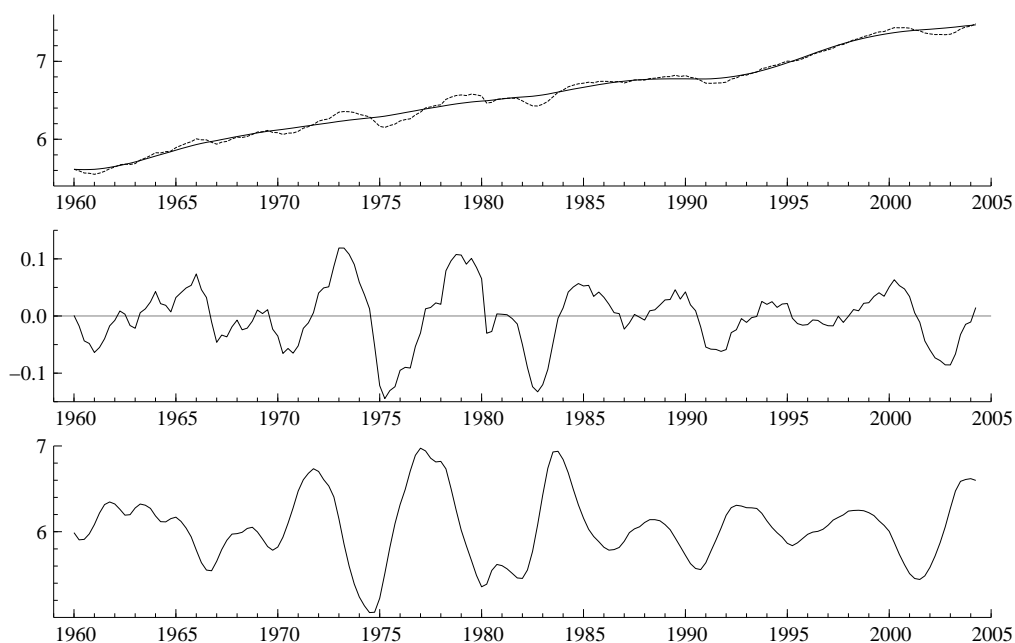

Figure 4: Trend-cycle decomposition of GDP.

The first plot shows the data and smoothed trend, the second plot shows the smoothed asymmetric cycle component, the third plot shows the cycle period.
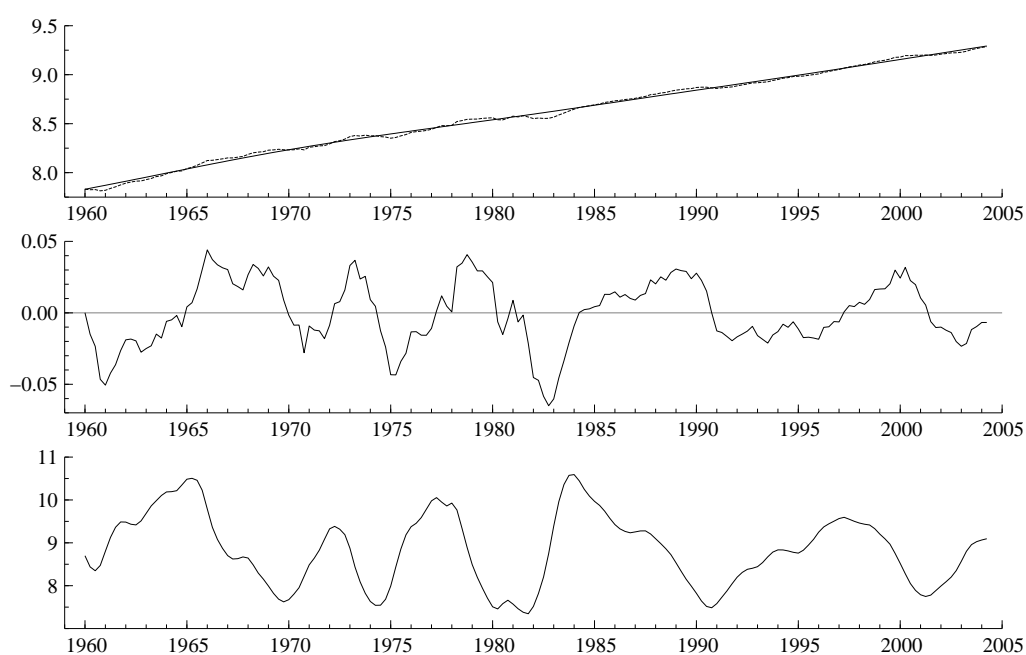
Figure 5: 1.000 largest importance weights in 100.000 simulation draws.

Importance weights are ratios of true and approximating densities and used as correction factors to the likelihood of an approximating model. A finite variance of the weights justifies the use of the importance sampling likelihood estimator as the central limit theorem applies.