Stationarity and ergodicity of new models for correlation dynamics

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- Stationarity and Ergodicity of dynamic dependence models
- Main challenge for stability conditions: Nonlinearity
  - No backwards subsitution scheme as in univariate GARCH or CCC-MGARCH analysis
  - Score is not bounded as in the univariate case (cf. Andres and Harvey (2012); Harvey and Sucarrat (2012))
  - Temporal dependence in scores as opposed to independent gamma or beta variates (dito)
- As in the univariate GAS case (cf. Blasques, Koopman and Lucas (2012)), use conditions put forward by Straumann and Mikosch (2006).

Setting

- Consider the scale model

$$
\begin{aligned}
y_t &= h(f_t;\lambda)u_t \\
f_{t+1} &= \omega(\theta) + \alpha(\theta)s_t(f_t;\lambda) + \beta(\theta)f_t.
\end{aligned}
$$

- $\{u_t\} \sim p_{u,\lambda}(u_t)$ i.i.d. with $\mathrm{E}_{t-1}[u_t] = 0$ and $\mathrm{Var}_{t-1}[u_t] = \mathrm{I}$
- A bivariate example:

$$
h(f_t;\lambda) = \begin{pmatrix} \sigma_{1,t}(f_t;\lambda) & 0 \\ 0 & \sigma_{2,t}(f_t;\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho(f_t;\lambda) & \sqrt{1-\rho(f_t;\lambda)^2} \end{pmatrix},
$$

where $\sigma_{i,t}(f_t;\lambda) = \exp(f_{i,t})$ for $i \in \{1,2\}$ and $\rho(f_t;\lambda) = \tanh(f_{3,t})$.

Represention of time-varying parameters

- $$f_{t+1} = \sum_{i=0}^{T} \beta^i \omega + \beta^{T+1} f_{t-T} + \sum_{i=0}^{T} \beta^i \alpha s_{t-i}(f_{t-i}; \lambda) \tag{1}$$
- Time-invariant if it exists as $T \to \infty$.
- If β has modulus greater than 1, the first term in (1) explodes as $T \to \infty$.
- The convergence of last term in (1) is hard to explicitly compute due to the nonlinearity inside s_t.
- Covariance stationarity: $\mathrm{E}[y_t] = 0$ and $\mathrm{E}[y_t y'_{t+i}] = 0$ for $i \neq 0$.

$\Rightarrow$ If elements of the variance matrix are affine in f_t, then $\{y_t\}$ is covariance stationary iff β has modulus less than 1.

Straumann and Mikosch (2006) revisited

- Stochastic recurrence equations approach:

$$f_{t+1} = \phi_t(f_t;\theta) := \omega(\theta) + \alpha(\theta)s_t(f_t;\lambda) + \beta(\theta)f_t \quad \forall\ t \in \mathbb{Z}.$$

- Linear folding if $s_t(f_t;\lambda) = \mathbb{W}(u_t)\, f_t$:

$$f_{t+1} = \Big(\alpha\mathbb{W}(u_{t-i}) + \beta\Big)^{T+1} f_{t-T} + \omega \sum_{i=0}^{T} \Big(\alpha\mathbb{W}(u_{t-i}) + \beta\Big)^{i}$$

- A Stationarity and Ergodicity constraint on (α,β) is

$$\mathrm{E}\Big[\log \|\alpha\mathbb{W}(u_t) + \beta\|\Big] < 0.$$

Straumann and Mikosch (2006) revisited

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$$f_{t+1} = \phi_t(f_t;\theta) := \omega(\theta) + \alpha(\theta)s_t(f_t;\lambda) + \beta(\theta)f_t \quad \forall\ t \in \mathbb{Z}.$$

- An essential Stationarity and Ergodicity constraint on θ is

$$\mathrm{E}\left[\log \sup_{(f,f^*)\in\mathcal{F}\times\mathcal{F}\ :\ f\neq f^*} \frac{\|\phi_t(f;\theta) - \phi_t(f^*;\theta)\|}{\|f - f^*\|}\right] < 0. \tag{2}$$

Constructive devices I

- Updating equation is in terms of y_t (cf. Creal, Koopman and Lucas (2011)), whereas stability conditions require **fixing randomness**.
- Consider correlation modeling by $y_t \sim \mathrm{N}(0, R(f_t;\lambda))$ with

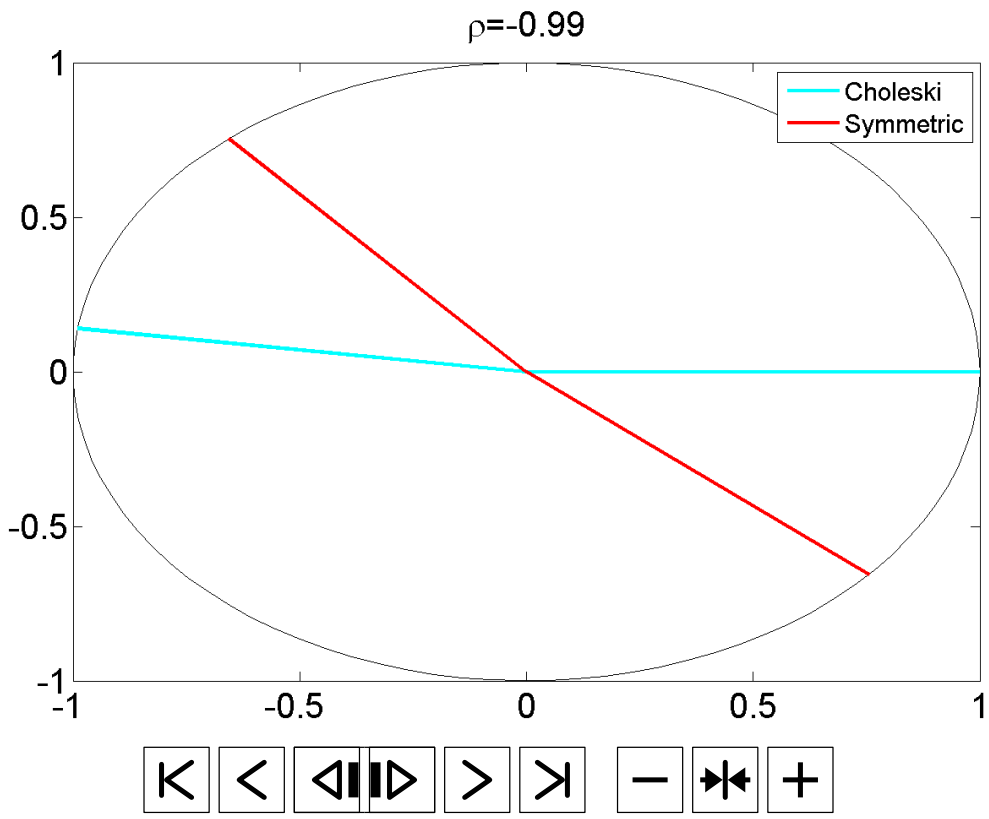
$$R(f_t;\lambda) = \begin{pmatrix} 1 & \rho(f_t;\lambda) \\ \rho(f_t;\lambda) & 1 \end{pmatrix},$$

- Let $u_t \sim \mathrm{N}(0, \mathrm{I})$. Two possible, **observationally equivalent** parametrizations for the scale matrix are

$$h(f_t;\lambda) = \begin{pmatrix} 1 & 0 \\ \rho(f_t;\lambda) & \sqrt{1-\rho(f_t;\lambda)^2} \end{pmatrix} \quad \text{(Choleski)}$$

$$h(f_t;\lambda) = \begin{pmatrix} \frac{1}{2}(\sqrt{1+\rho(f_t;\lambda)} + \sqrt{1-\rho(f_t;\lambda)}) & \frac{1}{2}(\sqrt{1+\rho(f_t;\lambda)} - \sqrt{1-\rho(f_t;\lambda)}) \\ \frac{1}{2}(\sqrt{1+\rho(f_t;\lambda)} - \sqrt{1-\rho(f_t;\lambda)}) & \frac{1}{2}(\sqrt{1+\rho(f_t;\lambda)} + \sqrt{1-\rho(f_t;\lambda)}) \end{pmatrix} \quad \text{(Symmetric)}$$

Constructive devices II



Covering the space

- In principle, any decompostion $h(f_t;\lambda) = (v_1, v_2)'$, where $v_i = (\cos(\alpha_i), \sin(\alpha_i))'$ of $R(f_t;\lambda)$ is possible as long as:
  - $\|v_1\| = \|v_2\| = 1$,
  - $\cos(\alpha_1 - \alpha_2) = \rho$.
- The following statement helps us limit decompositions that are relevant for different SE regions:

Proposition

Let $\tilde{h}(f_t;\lambda) = h(f_t;\lambda)Q$, where $Q = Q(f_t;\lambda)$ is an f_t-independent orthogonal matrix. If u_t is Gaussian, the implied SE regions for $h(f_t;\lambda)$ and $\tilde{h}(f_t;\lambda)$ are identical.

What does a typical link look like?

$$f_{t+1} = \omega + \beta f_t + \alpha \frac{\sqrt{1-\rho(f_t;\lambda)^2}\, u_{1,t} u_{2,t} - \rho(f_t;\lambda)(u_{2,t}^2 - 1)}{\sqrt{1+\rho(f_t;\lambda)^2}}$$

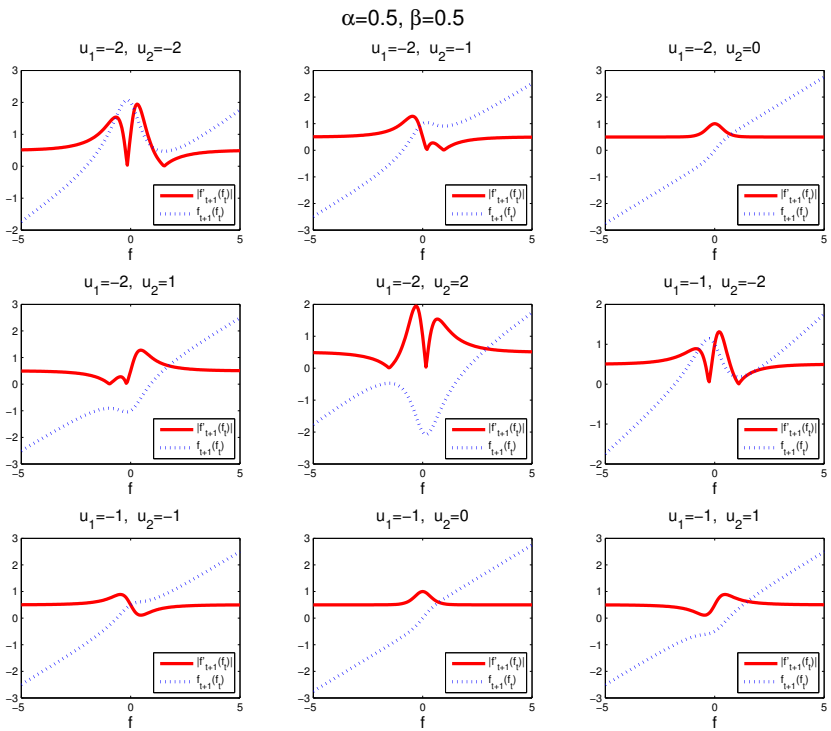


Figure 1: Shapes of $|\beta + \alpha \dot{s}_t(f;\lambda)|$ as a random function of f

- Two-step supremum evaluation:

Definition

- *Extremal score:* $\dot{s}_t^{(*)} = \dot{s}_t(f_t^{(*)}, \alpha, \beta, u, \psi)$, *where* $f_t^{(*)} = f_t^{(*)}(\alpha, \beta, u, \psi) = \arg\sup_f |\beta + \alpha \dot{s}_t'(f, u_t)|$.
- *Upward extremal score:* $s_{\sup}^{('*)} := \sup_f(\dot{s}_t(f, u))$
- *Downward extremal score:* $s_{\inf}^{('*)} := \inf_f(\dot{s}_t(f, u))$.

Remark

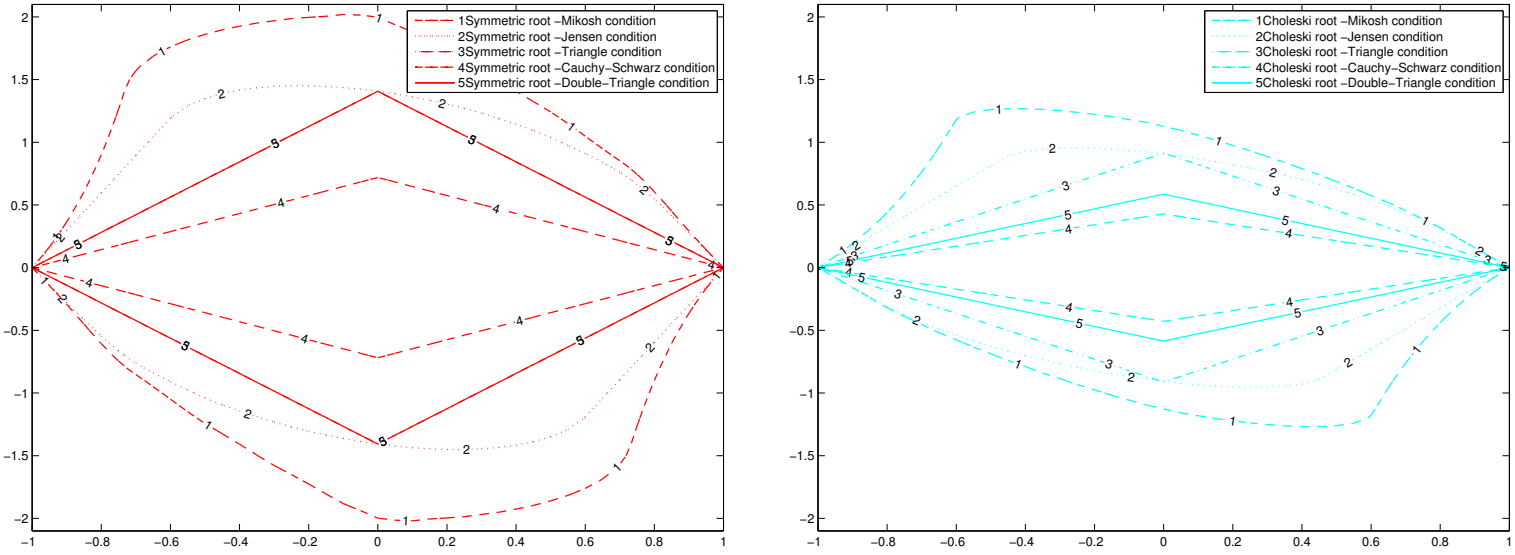
Calculation of the extremal score can be reduced to

$$\Lambda(\alpha, \beta)(\alpha, \beta) = E \log \sup_f |\beta + \alpha \dot{s}_t^{(*)}| = E \log \sup_f (|\beta + \alpha s_{\sup}^{('*)}|, |\beta + \alpha s_{\inf}^{('*)}|).$$

- Separates the effect of score from the static parameters (α, β)
- Computationally efficient

Effect of constructive device

Analytics



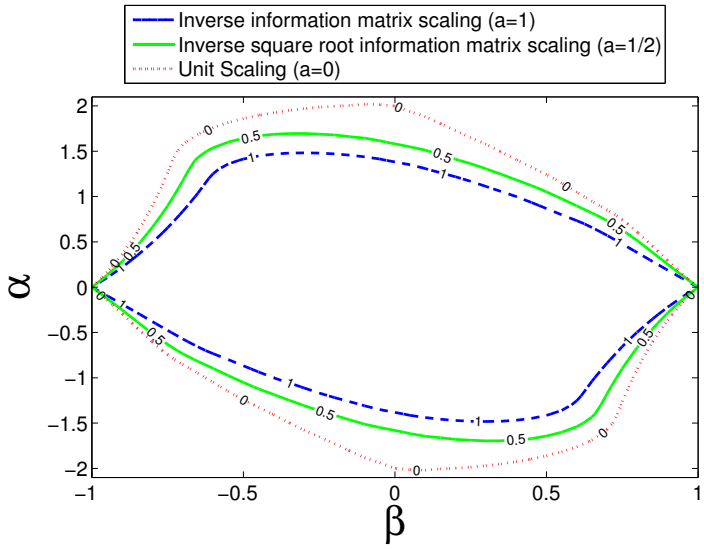
(a) (b)

Figure 2: Stationarity and Ergodicity sufficiency regions for unit scaling ($a = 0$) and different matrix decompositions

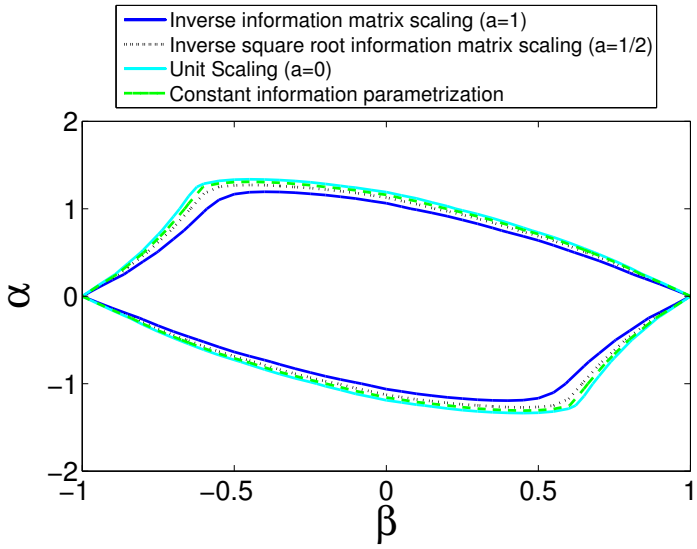
Properties

Different models

Fat tails



(a) Symmetric root



(b) Choleski root

Figure 3: Stationarity and Ergodicity sufficiency regions for different bivariate correlation models

Properties

Higher dimensional models

- Dynamic equicorrelation of Kelly and Engle (2012) has

$$R(f_t;\lambda) = (1-\rho(f_t;\lambda))\,\mathrm{I}_n + \rho(f_t;\lambda)\iota\iota',$$

- The score has

$$\nabla_t(f_t;\lambda) = \rho_3(f_t;\lambda)\left(-u_t'\nabla_{p_{u,\lambda}}(u_t) - n\right) + \rho_4(f_t;\lambda)\left(-\iota'\nabla_{p_{u,\lambda}}(u_t)u_t'\iota - n\right),$$

  where $\rho_3(f_t;\lambda)$ and $\rho_4(f_t;\lambda)$ are two independent functions

  $\rightarrow$ The score cannot be scaled to be independent of f_t.

- Unit scaling (S=1):

$$\dot{\nabla}_t(f_t;\lambda) = -\frac{n-1}{n}\dot{\rho}(f_t;\lambda)\left(-u_t'\nabla_{p_{u,\lambda}}(u_t) - n\right):$$

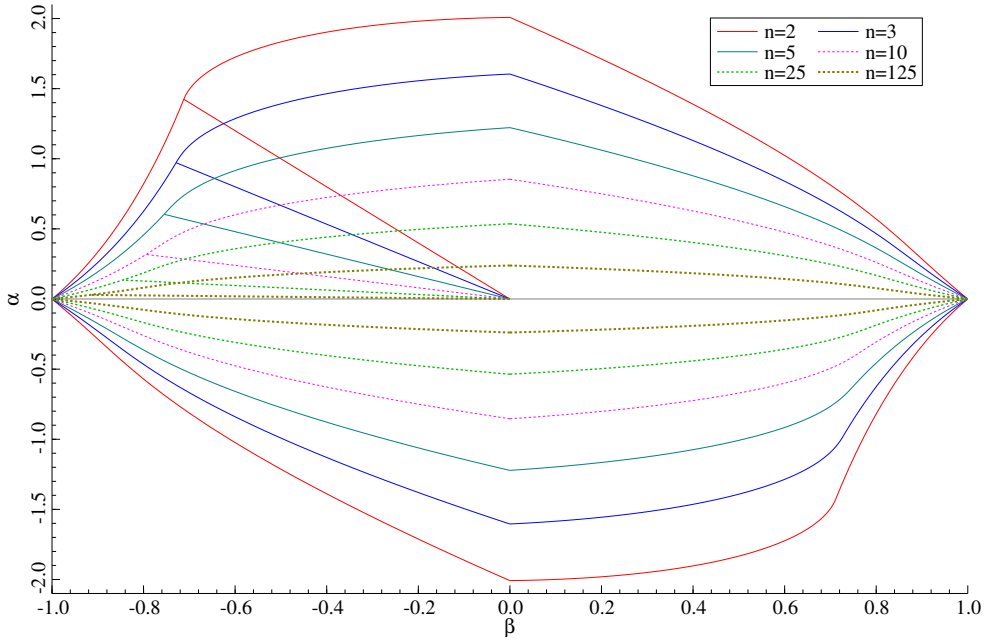


Figure 4: Stationarity and Ergodicity regions for the multivariate Gaussian equi-correlation model in *n* dimensions by the symmetric correlation matrix decomposition.

Main results

- Stationarity and ergodicity properties depend on the constructive device (covered by symmetric matrix root)
- Covariance stationarity is typically easier to characterize than strict stationarity
- Fat tails and cross-sectional dimension diminish the Straumann-Mikosch region, for which SE can be guaranteed

Separation property

Scale model has the density

$$\begin{aligned}
\log p_y(y_t|f_t;\lambda) &= \log p_{u,\lambda}(h(f_t;\lambda)^{-1}y_t) - \log|\det(h(f_t;\lambda))| \\
\Rightarrow \nabla(f_t;\lambda) &= \Psi(f_t;\lambda)'\mathrm{vec}\left(-\nabla_{p_{u,\lambda}}(u_t)u_t' - \mathrm{I}\right) \\
\Rightarrow \mathcal{I}_t(f_t;\lambda) &= \Psi(f_t;\lambda)'\left(\mathcal{I}_{p_{u,\lambda}} - \mathrm{vec}(\mathrm{I})\mathrm{vec}(\mathrm{I})'\right)\Psi(f_t;\lambda),
\end{aligned}$$

where

$$\begin{aligned}
\Psi(f_t;\lambda) &= \left(\mathrm{I}\otimes h(f_t;\lambda)^{-1}\right)\partial\mathrm{vec}(h(f_t;\lambda))/\partial f_t' \\
\mathcal{I}_{p_{u,\lambda}} &= \mathrm{E}[u_t u_t' \otimes \nabla_{p_{u,\lambda}}(u_t)\nabla_{p_{u,\lambda}}(u_t)'].
\end{aligned}$$

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Analytical expression for the derivative of the score

- It is sufficient to consider the class of observationally equivalent decomposition matrices, parametrized by

$$h(f_t;\lambda) = \begin{pmatrix} \cos(\phi(\rho(f_t;\lambda))) & \sin(\phi(\rho(f_t;\lambda))) \\ \sin(\psi(\rho(f_t;\lambda))) & \cos(\psi(\rho(f_t;\lambda))) \end{pmatrix}$$

  with $\phi(\rho) = \arcsin(\rho) - \psi(\rho)$ and $\psi(\rho) \in C^1(\mathbb{R},\mathbb{R})$.
- The first derivative of the score is given by

$$\begin{aligned} \dot{s}_\psi &= \sqrt{1-\rho^2}(\dot{\psi}\sqrt{1-\rho^2} - \frac{1}{2})z(\rho,\psi) - \frac{1}{2}(1-\rho^2)(u_1^2 + u_2^2 - 2), \\ z(\rho,\psi) &:= (u_1^2 - u_2^2)\rho\sin(2\psi) + 2u_1u_2\rho\cos(2\psi) \\ &\quad - \sqrt{1-\rho^2}\left(2u_1u_2\sin(2\psi) - (u_1^2 - u_2^2)\cos(2\psi)\right). \end{aligned}$$

- $\psi(\rho) := \arcsin(\rho) \leftrightarrow$ (Choleski)
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Fat Tails

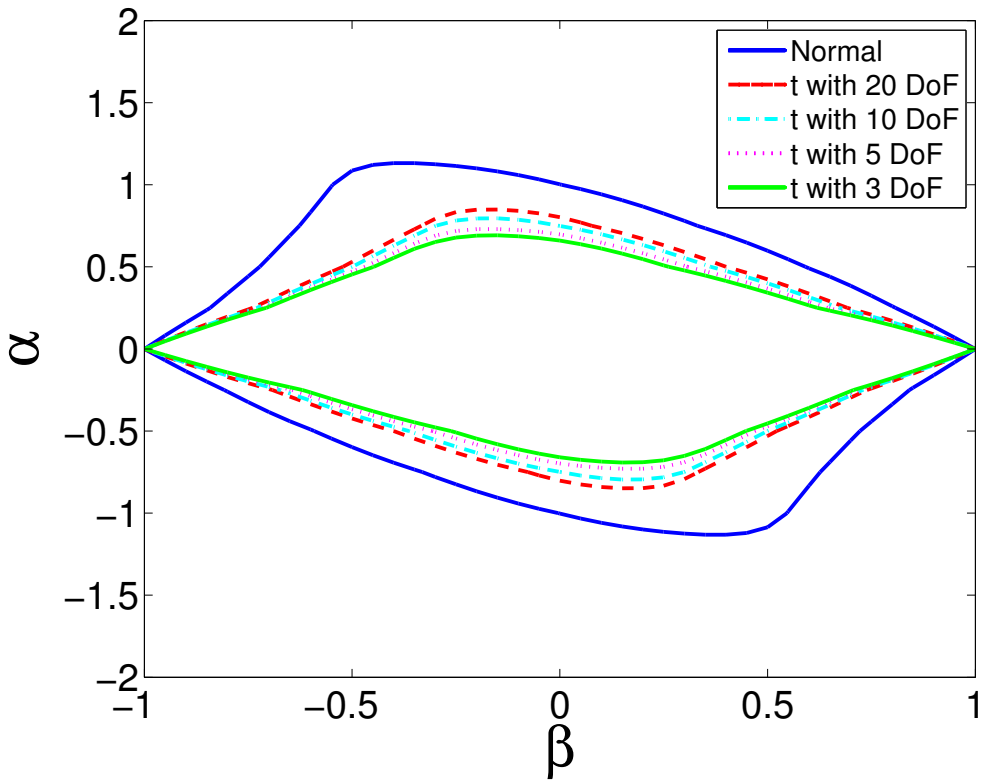

Figure 5: Stationarity and Ergodicity sufficiency regions for t_ν-distributed errors (Choleski decomposition)

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Characterizing the Straumann-Mikosch region

We observe

- Monotonicity (Uniqueness) of $\Lambda(\alpha, \beta)$ in the direction $h = (\theta \cdot \text{sign}(\alpha), \sqrt{1-\theta^2} \cdot \text{sign}(\beta))'$, where $\theta \in [0, 1]$
- Asymmetry
- Convexity after using Jensen's inequality
- Dimensionality diminishes the SE sufficiency region

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Semi-analytic expression distributional driver

Define G as the distribution function of $-u_t'\nabla_{p_{u,\lambda}}(u_t)$, which in the Gaussian case is $\chi^2(n)$:

- If $\alpha, \beta > 0$:

$$\mathrm{E}\sup_{f^*\in\mathbb{R}} |\beta + \alpha\dot{\nabla}_t(f^*;\lambda)| = \int_{-\infty}^{n}\left(\beta - \frac{1}{2}\alpha\,(x-n)\right)\,\mathrm{d}G(x) + \int_{n}^{n+4\beta/\alpha}\beta\,\mathrm{d}G(x) + \int_{n+4\beta/\alpha}^{\infty}\left(\frac{1}{2}\alpha\,(x-n) - \beta\right)\,\mathrm{d}G(x).$$

- If $\alpha < 0, \beta > 0$:

$$\mathrm{E}\sup_{f^*\in\mathbb{R}} |\beta + \alpha\dot{\nabla}_t(f^*;\lambda)| = \int_{-\infty}^{n+4\beta/\alpha}\left(\frac{1}{2}\alpha\,(x-n) - \beta\right)\,\mathrm{d}G(x) + \int_{n+4\beta/\alpha}^{n}\beta\,\mathrm{d}G(x) + \int_{n}^{\infty}\left(\beta - \frac{1}{2}\alpha\,(x-n)\right)\,\mathrm{d}G(x).$$

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Skewness

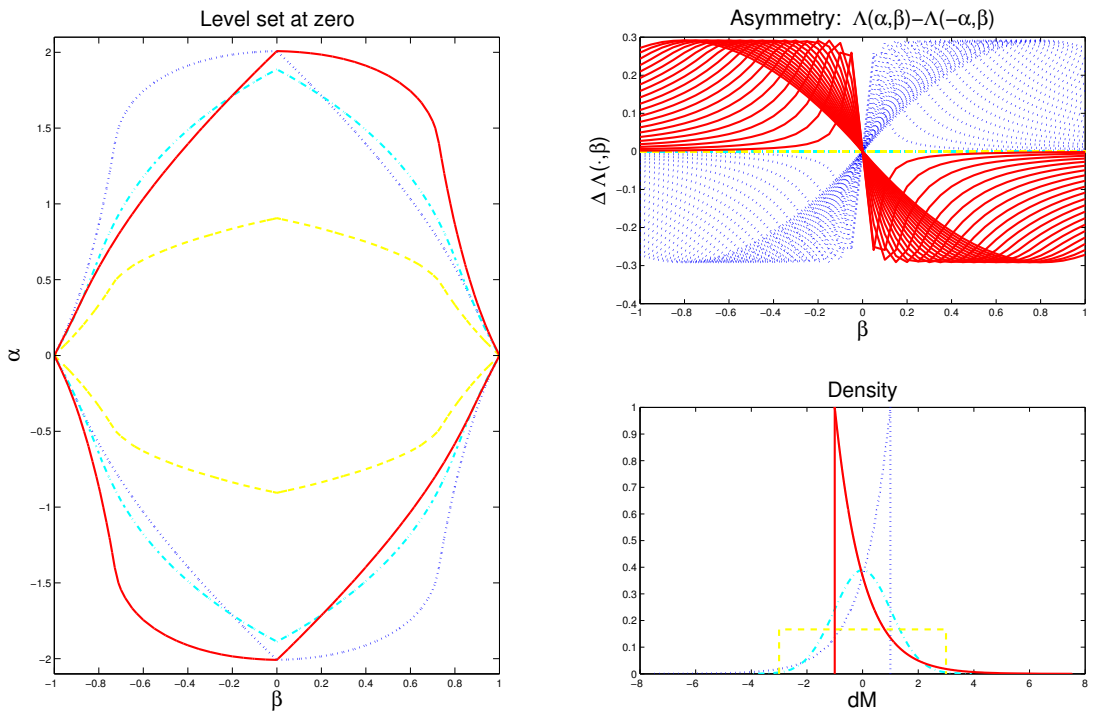

Figure 6: The effect of distributional asymmetry in the extremal score on the SE region ▸ Back