A Dynamic Model for Daily Equity Covariances Based on Multiple Measures

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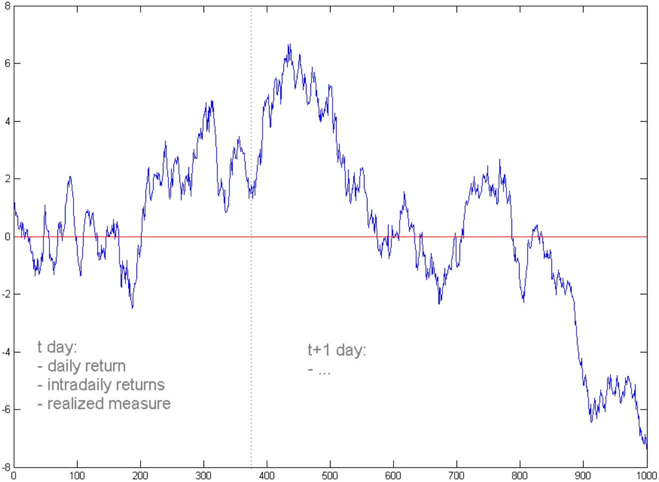
t day:
- daily return
- intradaily returns
- realized measure
t+1 day:
- ...
8
6
4
2
0
-2
-4
-6
-8
0
100
200
300
400
500
600
700
800
900
1000

The need for joint dynamics in returns and their dependency structure

Increasing availability of intraday financial data can lead to the development of more accurate forecasting models for the conditional covariance of daily returns :

- univariate
  - model for three latent volatility processes:
    Engle and Gallo (2006)
  - joint model for daily returns and realized volatility:
    Shephard and Sheppard (2010), Brownlees and Gallo (2010),
    Hansen et al. (2011)
- multivariate
  - parsimonious models for realized measures:
    Andersen et al. (2003), Voev (2008),
    Chiriac and Voev (2011) and Bauer and Vorkink (2011)
  - joint model for daily returns and realized covariance:
    Noureldin et al. (2011)

The contributions of this paper are :

- **simultaneous modeling** of returns and their covariance structure in a coherent manner
- potential use of **multiple measures** of realized covariance
- **parsimonious model** while still allowing for **cross-assets** effects or volatility spillover effects
- **straightforward estimation** of parameters in high-dimensional model
- **generalized autoregressive score** modeling framework

■ Observation model and parameter updating mechanism :

$$y_t \sim p(y_t|Y_{t-1}, f_t; \theta), \qquad t = 1, 2, \ldots, T, \tag{1}$$

$$f_{t+1} = \omega + \beta f_t + \alpha s_t, \tag{2}$$

where:

- y_t denotes dependent variable; $Y_t = [y_1, \ldots, y_t]'$
- f_t is the time-varying parameter of interest
- θ collects static parameters
- s_t is a **scaled score** based on $\partial \ln p(y_t|Y_{t-1}, f_t; \theta)/\partial f_t$.

We consider $k \geq 1$ assets and assume conditional densities:

$$\begin{array}{rll} \textit{daily returns}: & r_t|\mathcal{F}_{t-1} \sim \mathcal{N}_k(0, P_t), & \text{Normal} \\ \textit{realized measure}: & R_t|\mathcal{F}_{t-1} \sim \mathcal{W}_k(\nu, V_t/\nu), & \text{Wishart} \end{array}$$

where

$$\mathcal{N}_k = \frac{1}{(2\pi)^{\frac{k}{2}}|P_t|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}\, tr\left(P_t^{-1} r_t r_t'\right)\right\},$$

and

$$\mathcal{W}_k = \frac{|R_t|^{\frac{\nu-k-1}{2}}}{2^{\frac{\nu k}{2}}\nu^{-\frac{\nu k}{2}}|V_t|^{\frac{\nu}{2}}\Gamma_k\left(\frac{\nu}{2}\right)} \exp\left\{ -\frac{\nu}{2}\, tr\left(V_t^{-1} R_t\right)\right\},$$

with degrees of freedom $\nu \geq k$ and multivariate Gamma function

$$\Gamma_k\left(\frac{\nu}{2}\right) = \pi^{\frac{k(k-1)}{4}} \prod_{i=1}^{k} \Gamma\left(\frac{\nu}{2} + (1-i)/2\right).$$

The link between the Normal and Wishart densities can be established naturally via the relation

$$P_t = \Lambda^{1/2} V_t \Lambda^{1/2}.$$

Here $\Lambda^{1/2}$ can capture a correction for overnight variation if returns r_t are close-to-close.

When returns are open-to-close, Λ can be close to an identity matrix.

The score vector takes an additive form given by

$$\nabla_t = \sum_{i=1}^{m} \nabla_{i,t} = \sum_{i=1}^{m} \frac{\partial \ln p_i(y_t^i | \mathbf{Y}_{t-1}, f_t; \theta)}{\partial f_t},$$

which corresponds to the sum of individual scores. The scaling term is based on the individual information matrices as given by

$$\mathcal{I}_t = \sum_{i=1}^{m} \mathcal{I}_{i,t} = \sum_{i=1}^{m} \mathbb{E}[\nabla_{i,t} \nabla'_{i,t} | \mathcal{F}_{t-1}].$$

It leads to a flexible modeling framework for time series with different characteristics; see Creal, Schwaab, K&L (2011) for an application in credit risk.

We define the dynamics for vechtorized Cholesky decomposition:

$$f_t = vech(C_t), \quad V_t = C_t C_t', \quad V_t = unvech(f_t)\, unvech(f_t)'.$$

Proposition

For our model, score vector has dimension $\big(k(k+1)/2\big) \times 1$:

$$\nabla_t = \frac{1}{2}\dot{V}_t' D_k' \times$$

$$\big(V_t^{-1} \otimes V_t^{-1}\big)\Big(\nu\big[vec(R_t) - vec(V_t)\big] + \big[vec(r_t r_t') - vec(V_t)\big]\Big),$$

$$\mathcal{I}_t = \mathbb{E}[\nabla_t \nabla_t^T | \mathcal{F}_{t-1}] = \frac{1+\nu}{4}\dot{V}_t' D_k' \big(V_t^{-1} \otimes V_t^{-1}\big)\big(I_{k^2} + K_k\big) D_k \dot{V}_t.$$

where $\otimes$ is Kronecker product, D_k is duplication matrix, K_k is commutation matrix, $\dot{V}_t$ is derivative of Cholesky decomposition.

For scaled score, focus is on the term

$$\Big(\nu\big[\text{vec}(R_t) - \text{vec}(V_t)\big] + \big[\text{vec}(r_t r_t') - \text{vec}(V_t)\big]\Big).$$

- The updating mechanism for covariance process utilizes two (possibly more) realized measures.
- Since $\nu \geq k$ the main driving force is $\nu\big[\text{vec}(R_t) - \text{vec}(V_t)\big]$ while relatively less information is taken from $r_t r_t'$. This is in contrast to BEKK which purely relies on $r_t r_t'$ realized measure.
- Each element of ∇_t exploits the full log-likelihood information. Hence for any model specification, including the **scalar** formulation $f_{t+1} = \omega + \beta f_t + \alpha s_t$, cross-asset effects are incorporated. This is in contrast to scalar BEKK and scalar HEAVY where the cross-asset effects have vanished. We regard this as an important result !

- r_t daily return, R_t realized variance, V_t true latent daily variance, we obtain

$$r_t | \mathcal{F}_{t-1} \sim N(0, V_t), \qquad R_t | \mathcal{F}_{t-1} \sim Gamma(\nu, V_t/\nu).$$

- To ensure the variance process be positive at all times, we model **log-variance** defined as $f_t = \log V_t$ to obtain

$$\nabla_t = \frac{1}{2V_t}\Big(\nu (R_t - V_t) + (r_t^2 - V_t) \Big) \qquad \text{and} \qquad \mathcal{I}_t = \frac{1+\nu}{2}.$$

- In this way we obtain a new realized (E)GARCH model

$$f_{t+1} = \omega + \beta f_t + \alpha \left\{ \nu \left(\frac{R_t}{V_t} - 1 \right) + \left(\frac{r_t^2}{V_t} - 1 \right) \right\},$$

where daily variance is driven by sum of squared returns and realized variance measure, cf. Hansen, Huang & Howard (2011).

The primary focus is on the information extraction and the role of ν. Let us consider this setup :

- we simulate price path with n intraday observations, $n \in [5, 10, 15, 20, 30, 50, \ldots]$, for $T = 1000$ trading days
- the variance process is near random walk process and has its own source of error (GAS in not MC DGP)
- we compute daily returns r_t and realized variances R_t with n intraday observations
- there is no microstructure noise in this setup
- we estimate the new model in its scalar specification, $\theta = [\omega, \alpha, \beta, \nu]$
- we repeat the simulation & estimation M times

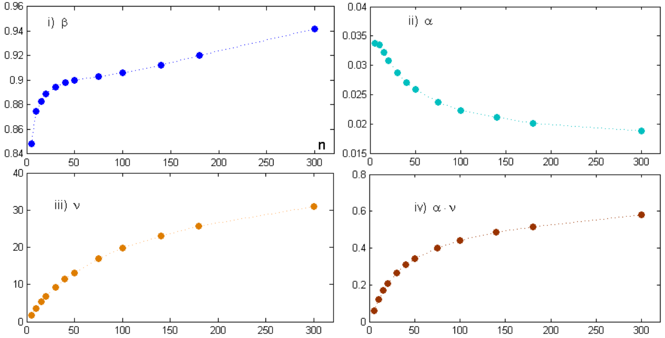

Figure 1: ML estimates of θ vs sampling frequency n

Recall the score expression

$$\alpha s_t \propto \alpha\left(\nu\left[\text{vec}(R_t) - \text{vec}(V_t)\right] + \left[\text{vec}(r_t r_t') - \text{vec}(V_t)\right]\right),$$

and note increasing relevance of R_t relative to $r_t r_t'$ as n increases; $\sqrt{n}(R_t - V_t) \rightarrow MN(0, 2IQ_t)$.

ML estimation is based on the BFGS algorithm (`Ox`) with numerical derivatives (the closed-form derivatives are at hand but for ν):

- the scaling with $S_t = \mathcal{I}_t^{-1/2}$ appears to be preferred one (which is a bit unfortunate from computational viewpoint as it entails inverse of a matrix of order $O(k^2)$ at each step ... due to this finding this step may call for some further computational improvements)
- the simulation density of the degrees of freedom in Wishart density behaves very well, which might be surprising given that model is highly nonlinear in this parameter
- the simulation density of AR coefficient β is left-skewed indicating tendency to underestimate the persistence of the processes
- but in the scalar model formulation, the increasing dimension k helps to reduce the (downward) bias of β

- We notice that

$$r_t|\mathcal{F}_{t-1} \sim \mathcal{N}_k(0, V_t) \qquad \rightarrow \qquad r_t r_t'|\mathcal{F}_{t-1} \sim \mathcal{SW}_k(1, V_t),$$

where $\mathcal{SW}_k$ denotes singular Wishart (df=1) with density

$$\frac{tr(r_t r_t')^{\frac{-k}{2}}}{(2\pi)^{\frac{k}{2}}|V_t|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}\, tr\left(V_t^{-1} r_t r_t'\right)\right\},$$

see Srivastava (2003).

- Then define the measurement densities as

$$\begin{aligned} \textit{outer product of daily returns}: &\quad r_t r_t'|\mathcal{F}_{t-1} \sim \mathcal{SW}_k(1, V_t), \\ \textit{realized measure}: &\quad R_t|\mathcal{F}_{t-1} \sim \mathcal{W}_k(\nu, V_t/\nu), \end{aligned}$$

we obtain the same updating equation for f_t.

Proposition

Consider multiple noisy measures of daily equity covariances:

$$R_t^i = V_t^{1/2} \eta_t^i V_t^{1/2}, \qquad \eta_t^i \sim \mathcal{W}_k(\nu^i, I_k/\nu^i),$$

where R_t^i is a noisy measure of daily realized covariance matrix, for $i = 1, \ldots, m$. Then

$$\nabla_t = \frac{1}{2} \dot{V}_t' D_k' \left(V_t^{-1} \otimes V_t^{-1} \right) \left(\sum_{i=1}^{m} \nu^i \left[vec(R_t^i) - vec(V_t) \right] \right),$$

$$\mathcal{I}_t = \mathbb{E}[\nabla_t \nabla_t^T | \mathcal{F}_{t-1}] = \dot{V}_t' D_k' \left(V_t^{-1} \otimes V_t^{-1} \right) \left(I_{k^2} + K_k \right) D_k \dot{V}_t \, \frac{\sum_{i=1}^{m} \nu^i}{4},$$

with $\nu^1, \nu^2, \ldots, \nu^m$ as df's (notice $\nu^i \equiv 1$ if $R_t^i = r_t r_t'$).

We evaluate the performance of the new modeling framework against GARCH and EWMA:

- 15 equities (AA AXP BA CAT GE HD HON IBM JPM KO MCD PFE PG WMT XOM) from TAQ through WRDS
- different dimensions: 2×2, 5×5 and "full" model with dimension 15×15 (dimension restriction due to RFS and not due to model !)
- data cleaned following guidelines of BNHLS (2008)
- we consider the score model with different scalings
- we apply close-to-close and close-to-open returns separately to quantify overnight variation
- different realized measures: subsampled 5min realized covariance, 15min realized covariance, BNHLS Kernel-based realized covariance
- we evaluate the models using one-step ahead forecasts using RMSE and quasi-likelihood loss function

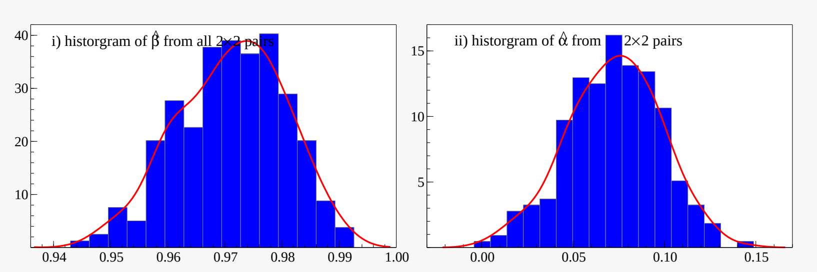

Figure 2: Histograms β and α coefficients (close-to-open returns), 2×2 case

- β estimates range 0.97-0.98 (estimate in 15×15 case: 0.9828)
- α estimates range 0.05-0.10 (estimate in 15×15 case: 0.0268)
- similarities of β and α estimates support scalar model: does not rule out cross-equity effects
- to pool/group β's and α's, by risk domicile countries or by industries, etc.

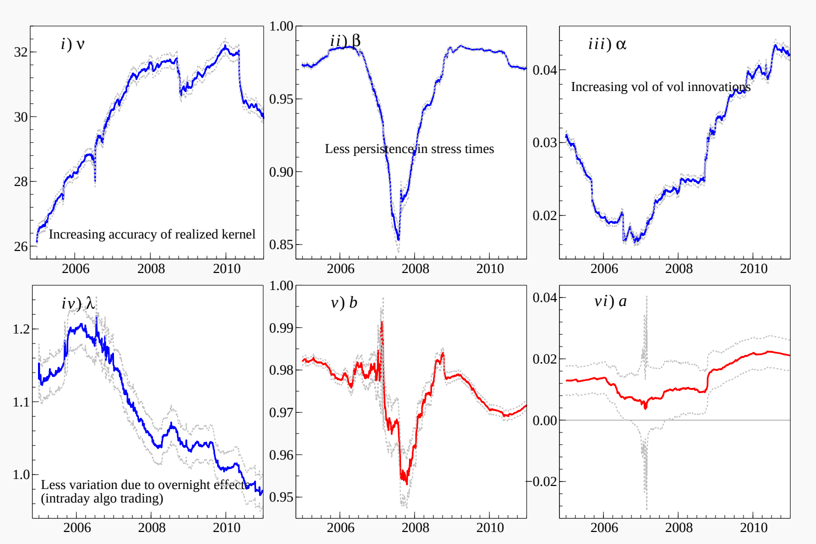

Figure 3: Panels i) to iv) score model, panel *v*) and *vi*) GARCH-BEKK for the 15×15 scalar case.

Summary of deliverables :

- we have developed a dynamic model for both multiple returns and realized measures of covariance
- it enables signal extraction of covariance structure from multiple measures (different frequency, different robustness properties, ...)
- it allow for cross-asset effects with a small number of static parameters
- it enables to extract daily time-varying "*betas*" as measures of systematic risk
- is based on stable and manageable estimation routines
- we can add other risk factors, e.g. oil shocks to the equity universe