Sums and products

BUSINESS MATHEMATICS
CONTENTS

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DEFINITION OF SUMMATION

Is there a quick way to write $1 + 2 + \cdots + 9 + 10$? Yes, use the **summation operator** $\sum$ as follows:

$$\sum_{i=1}^{10} i$$

More general, we define:

$$\sum_{i=m}^{n} x_i = x_m + x_{m+1} + \cdots + x_{n-1} + x_n$$
DEFINITION OF SUMMATION

- Example 1: take \( x_i = i \), \( m = 1 \), and \( n = 10 \):
  \[
  \sum_{i=m}^{n} x_i = \sum_{i=1}^{10} i = 1 + 2 + \cdots + 10 = 55
  \]

- Example 2: take \( x_i = 2 \) (for all \( i \)), \( m = 0 \), and \( n = 10 \):
  \[
  \sum_{i=m}^{n} x_i = \sum_{i=0}^{10} 2 = 2 + 2 + \cdots + 2 = 22
  \]
DEFINITION OF SUMMATION

Notice the alternative use of display form and inline form:

\[ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i \]

When unambiguous you may simplify:

\[ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i = \sum x \]

Never use the index variable as lower or upper boundary:

\[ \sum_{i=i}^{n} x_i \quad \text{or} \quad \sum_{i=1}^{i} x_i \]
DEFINITION OF SUMMATION

Index symbol “disappears”:

\[
\sum_{i=1}^{n} x_i = q
\]

\[
\sum_{i=1}^{n} x_{ij} = q_j
\]

Index symbol is arbitrary:

\[
\sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j = \sum_{\alpha=1}^{n} x_\alpha
\]
EXERCISE 1

Find $\sum_{i=0}^{3} (i^2)$ and $(\sum_{i=0}^{3} i)^2$
MOTIVATION

Why use $\Sigma$?

- in business analyses, we often need to add over years, over departments, etc.
- it helps us to prepare for later subjects (statistics, matrices)
- it is important in applied courses (finance, accounting)
**EXEMPLARY**

- Each year $t$ (starting at $t = 0$), you deposit an amount $A_t$ euro at a bank account (at 0% interest)
- What is the amount after $n$ years?

**Solution**

- Define the amount at the end of year $t$ as $x_t$
- Clearly at the start moment $t = 0$, we have $x_0 = A_0$
- At $t = 1$, we have $x_1 = x_0 + A_1 = A_0 + A_1$
- At $t = 2$, we have $x_2 = x_1 + A_2 = A_0 + A_1 + A_2$
- At $t = n$, we have $x_t = \sum_{t=0}^{n} A_t$
PROPERTIES

Additivity: \[ \sum (x_i + y_i) = \sum x_i + \sum y_i \]

Homogeneity: \[ \sum cx_i = c \sum x_i \]

Extending terms: \[ \sum_{i=1}^{n} x_i + \sum_{i=n+1}^{n+m} x_i = \sum_{i=1}^{n+m} x_i \]

Constant term: \[ \sum_{i=1}^{n} c = nc \]

Telescopic series: \[ \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 \]
Most are easy to see

- e.g.,

\[
\sum_{i=1}^{4} cx_i = cx_1 + cx_2 + cx_3 + cx_4 \\
= c(x_1 + x_2 + x_3 + x_4) \\
= c \sum_{i=1}^{4} x_i
\]

- but we will skip most proofs
EXERCISE 2

Find $\sum_{i=1}^{5} (7 - i)$
EXERCISE 3

Given $x_i = 2y_i + 1$, find $\sum_{i=1}^{3}(x_i + 2)$
DOUBLE SUMMATION

Straightforward to generalize:
\[
\sum_{i=1}^{n} \left( \sum_{j=1}^{m} x_{ij} \right) = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} x_{ij} \right)
\]
therefore no ambiguity in writing \( \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \)

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = x_{11} + \cdots + x_{1m} + x_{21} + \cdots + x_{2m} + \cdots + x_{nm}
\]

- Example: take \( x_{ij} = i + 2j \) and \( n = 4 \) and \( m = 3 \):
- \( \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = (1 + 2) + (1 + 4) + \cdots + (4 + 6) = 78 \)
EXERCISE 4

Find $\sum_{k=1}^{3} \sum_{j=1}^{k-1} j$
We also define the **product operator** $\prod$ as follows:

\[ \prod_{i=m}^{n} x_i = x_m \times x_{m+1} \times \cdots \times x_{n-1} \times x_n \]

- **example 1**: take $x_i = i$, $m = 1$, and $n = 10$:
  - $\prod_{i=m}^{n} x_i = \prod_{i=1}^{10} i = 1 \times 2 \times \cdots \times 10 = 10! = 3628800$

- **example 2**: take $x_i = 2$, $m = 0$, and $n = 10$:
  - $\prod_{i=m}^{n} x_i = \prod_{i=0}^{10} 2 = 2 \times 2 \times \cdots \times 2 = 2^{11} = 2048$
EXAMPLE

- At year 0, you deposit an amount $A$ euro at a bank account
- At the end of year $t$, you receive $r_t \%$ interest
- What is the amount after $n$ years?

Solution

- Define the amount at the end of year $t$ as $x_t$
- Clearly at the start moment $t = 0$, we have $x_0 = A$
- At $t = 1$, we have $x_1 = x_0 \left(1 + \frac{r_1}{100}\right) = A \left(1 + \frac{r_1}{100}\right)$
- At $t = 2$, we have $x_2 = x_1 \left(1 + \frac{r_2}{100}\right) = A \left(1 + \frac{r_1}{100}\right) \left(1 + \frac{r_2}{100}\right)$
- At $t = n$, we have $x_t = A \prod_{t=1}^{n} \left(1 + \frac{r_t}{100}\right)$
EXERCISE 5

Find $\prod_{i=0}^{100} 3i$
EXERCISE 6

Find $\prod_{n=1}^{2} \sum_{j=1}^{n} j$
Given is the function $S(k) = \sum_{i=k}^{2k} (k + i)$. Compute $S(3)$. (formula or exact)
A customer card is introduced to increase turnover. After a customer’s $n^{th}$ visit, the discount at the next visit is $\sqrt[3]{10n}$ percent. Give a formula for the cumulative discount for a person who has had dinner $m$ times, and who always orders meals and drinks which have a full (undiscounted) price of 60€. (6 points)
FURTHER STUDY

Sydsæter et al. 5/E 2.8, 2.9, 2.11

Tutorial exercises week 1

sums, double sums, products