Counting Vertex Covers in General Graphs

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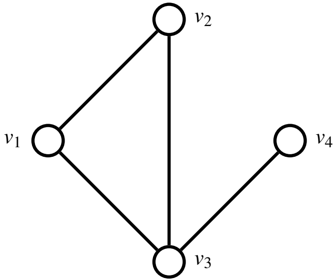
What is a Vertex Cover in a Graph?

- A set of vertices such that each edge of the graph is incident to at least one vertex of the set.
- Example



- Finding a minimum vertex cover is one of the classical NP-complete decision problems.
- $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are minimal vc's. All supersets of these are vc.

Associated Counting Problem

- How many vertex covers are there for a given graph?
- #P-complete counting problem.
- Related to propositional model counting.
- Efficient model counting algorithms are of interest for Bayesian inference problems or combinatorial design problems.

Randomized Approximation Algorithms

- We will consider simple undirected graphs $G = G(V, E)$.
- Let $c_G(n)$ be the exact (but unknown) number of vertex covers in an instance graph G with $n = |V|$ vertices.
- A *randomized algorithm* produces a random output $\widehat{c}_G(n)$ as estimate.
- A randomized algorithm is a *fully polynomial randomized approximation scheme* (FPRAS) if for every triple (n, ϵ, δ) the output satifies

  $$\mathbb{P}\big((1-\epsilon)c_G(n) < \widehat{c}_G(n,\epsilon,\delta) < (1+\epsilon)c_G(n)\big) > 1-\delta$$

  in a running time that is polynomial in ϵ^{-1}, $\log\delta^{-1}$ and n.
- Note that ϵ and δ may be part of the input of the estimator.

FPRAS Successes

- Other combinatorial counting problems.
- Generally hard to construct FPRAS.
- Some (not exhausted!) are
  - Karp et al. (1989) for counting the number of satisfying assignments to a boolean formula in disjunctive normal form.
  - Jerrum and Sinclair (1996) for counting the number of matchings (of all sizes) in a graph.
  - Cryan and Dyer (2003) for the number of contingency tables when the number of rows is constant.
  - Dyer (2003) for counting the number of solutions to a 0-1 knapsack problem.
  - Jerrum et al. (2004) for counting the permanent of a matrix with nonnegative entries.

FPRAS for Counting Vertex Covers in a Graph

- Not (yet?) developed.
- But ...

FPRAS for Counting Vertex Covers in Random Graphs

We have constructed an algorithm that shows *FPRAS for random graphs*. This means

- Let $\mathscr{S}(n)$ be the set of all (simple undirected) graphs with n vertices.
- Then

$$\mathbb{P}_{\mathrm{EG}}(\text{algorithm is FPRAS for } G \in \mathscr{S}(n)) \to 1,$$

  as $n \to \infty$, when G is drawn randomly from $\mathscr{S}(n)$ according to the Edgar Gilbert model.

- This means that each edge from the $\binom{n}{2}$ possible edges is present with probability $1/2$.

The Algorithm

Importance sampling.

- Given an undirected simple graph $G = G(V, E)$ with $n = |V|$ vertices.
- Consider binary vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$.
- Any binary vector corresponds one-to-one with a vertex set $V(\mathbf{x}) \subset V$ by

$$v_i \in V(\mathbf{x}) \quad \Leftrightarrow \quad x_i = 1.$$

- Let f be a proposal PMF on $\{0, 1\}^n$ such that

$$V(\mathbf{x}) \text{ is vertex cover in } G \quad \Rightarrow \quad f(\mathbf{x}) > 0.$$

- Then

$$c_G(n) = \mathbb{E}_f \left[\frac{\mathbb{I}\{V(\mathbf{x}) \text{ is vertex cover in } G\}}{f(\mathbf{X})} \right].$$

Sequential Importance Sampling (SIS)

- Decomposition by conditional PMF's:

$$f(\mathbf{x}) = \prod_{i=1}^{n} f_i(x_i | x_1, \ldots, x_{i-1}).$$

- Given a proposal f,
  - easy to generate $x_1, x_2, \ldots$ iteratively from the conditional PMF's;
  - hence, easy to get binary vector $\mathbf{x} \overset{\mathcal{D}}{\sim} f$;
  - finally, easy to check vertex cover property of associated vertex set $V(\mathbf{x})$.
- Repeat N times to get unbiased estimator

$$\widehat{c}_G(n) = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathbb{I}\{V(\mathbf{X}_i) \text{ is vertex cover in } G\}}{f(\mathbf{X}_i)},$$

- For what proposal f is SIS algorithm FPRAS for random graphs?

The Zero-variance Proposal PMF

- Define
$$f^*(\mathbf{x}) = \frac{1}{c_G(n)}\,\mathbb{I}\{V(\mathbf{x}) \text{ is vertex cover in } G\}.$$
- Then $\mathbb{V}ar_{f^*}(\widehat{c}_G(n)) = 0$.
- This is optimal importance sampling (and certainly FPRAS).
- Unfortunately, not implementable.
- But ...

Decomposition of Zero-variance PMF

- We can show that

$$f^*(\mathbf{x}) = \prod_{i=1}^{n} f_i^*(x_i | x_1, \ldots, x_{i-1}).$$

- Where

$$f_i^*(1 | x_1, \ldots, x_{i-1}) = \frac{c_{G^{[i]}}}{c_{G^{[i]}} + c_{G^{[-i]}}}$$

$$f_i^*(0 | x_1, \ldots, x_{i-1}) = 1 - f_i^*(1 | x_1, \ldots, x_{i-1})$$

- Where
  - $G^{[i]}$ and $G^{[-i]}$ are specific (known) subgraphs of G, given by the values of $x_1, \ldots, x_{i-1}$;
  - $c_{G^{[i]}}$ is the associated number of vertex covers in subgraph $G^{[i]}$ (exact but unknown).

An Implementable Proposal PMF

- Approximate the conditional zero-variance PMF's:

$$f_i(1|x_1,\ldots,x_{i-1}) = \frac{A^{[i]}}{A^{[i]} + A^{[-i]}}.$$

- Where $A^{[i]}$ and $A^{[-i]}$ are computable approximations of $c_{G^{[i]}}$ and $c_{G^{[-i]}}$, respectively.
- As follows (for $c_{G^{[i]}}$):
  - Given $x_1,\ldots,x_{i-1}$, determine subgraph $G^{[i]}$;
  - Say $G^{[i]}$ has k vertices;
  - Let $\mathscr{G}$ be a random graph of k vertices according to the Edgar Gilbert model;
  - Then set $A^{[i]} = \mathbb{E}_{\text{EG}}[c_{\mathscr{G}}(k)]$;
- Easy to compute

$$\mathbb{E}_{\text{EG}}[c_{\mathscr{G}}(k)] = \sum_{i=0}^{k} \binom{k}{i} 2^{-\binom{i}{2}}$$

Main Result

Theorem

The SIS algorithm with the approximated conditional zero-variance PMF's is FPRAS for counting vertex covers in random graphs.

The proof is based on a similar result for counting cliques (Rasmussen 1997) and the relation between vertex covers in a graph and cliques in the complement graph.

Improved Algorithm

- Again approximate the vertex cover numbers $c_{G^{[i]}}$ and $c_{G^{[-i]}}$ that pop up in the expression of the conditional zero-variance PMF's:

$$\widetilde{f}_i(1|x_1,\ldots,x_{i-1}) = \frac{B^{[i]}}{B^{[i]}+B^{[-i]}}.$$

- Approximation is based on a vertex cover relaxation.

Vertex Cover Relaxation

- Consider the subgraph $G^{[i]} = (V^{[i]}, E^{[i]})$.
- Suppose k vertices.
- Label the vertices in some order $v_1, \ldots, v_k$.
- Define probabilities $p_i = d_i/(k-1)$;
- where $d_i =$ the number of downstream (i.e., $j > i$) neighbours of v_i.
- Define a probability space Ω_G of all graphs $G' = (V^{[i]}, E')$ with the same vertex set $V^{[i]}$;
- but where each possible edge $(v_i, v_j),\ j > i$ is present in E' with probability p_i.
- Let $\mathscr{G}$ be a random graph in this probability space.
- Then set $B^{[i]} = \mathbb{E}_{\Omega_G}[c_{\mathscr{G}}(k)]$.
- It can be shown that the computation of $B^{[i]}$ has polynomial complexity $\mathcal{O}(k^2)$.
- Similarly for the subgraph $G^{[-i]}$.

Comparison

Conjecture

The SIS estimators of counting vertex covers satisfy

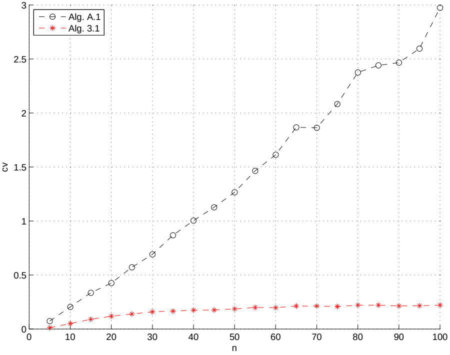
$$\mathbb{V}ar_{\widetilde{f}}(\widehat{c}_G(n)) \leq \mathbb{V}ar_f(\widehat{c}_G(n)).$$

Experiments

- Our SIS algorithms denoted *Alg. A* and *Alg. B*.
- *Cachet* is exact model counting software introduced by Sang et al. (2004); based on a SAT solver.
- *SampleSearch* is a probabilistic model counting technique by Gogate and Dechter (2006, 2007); based on sampling from the search space of a Boolean formula.
- No randomized algorithms have been developed dedicated to the vertex cover counting problem.

Random Graphs

- 40 random graphs for each $n = 5, 10, \ldots, 100$.
- Plot of the estimated coefficients of variation of the SIS estimators (ratio of variance and square mean).



A Small Model

- $n = |V| = 100$ vertices and $|E| = 2432$ edges.
- Exact (Cachet): $c_G(n) = 244941$.
- Alg. B: estimate `2.444e+05` with (numerical) relative error `1.28e-02`.
- SampleSearch: estimate 196277!

A Large Model

- $n = |V| = 1000$ vertices and $|E| = 249870$ edges.
- Alg. B estimate `2.773e+11` with (statistical) relative error `1.579e-02`.
- Cachet and SampleSearch failed.

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