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APPROXIMATING SENSITIVE QUEUEING NETWORKS BY REVERSIBLE MARKOV CHAINS

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Abstract. A general method will be presented to approximate networks of queues for which the stationary distribution is not of the standard product form. The approximation is made through reversible Markov chains. This means that the obtained bounds for the stationary distribution are insensitive with respect to service time distributions. The results are derived by using the treeform solution of the stationary distribution of a Markov chain.

1. INTRODUCTION

It is well-known that for a reversible Markov chain the Kolmogorov property holds (cf. [6]), "the product of the transition rates in a cycle of states in one direction is equal to the product of the transition rates in the reversed direction". This property implies the following closed form expression for the stationary probabilities,

$$\frac{\pi_j}{\pi_i} = \frac{q_{ij_1} q_{j_1j_2} \cdots q_{j_{n-1}j}}{q_{j_1i} q_{j_2j_1} \cdots q_{jj_{n-1}}}, \tag{1.1}$$

with $q_{k\ell}$ the transition rate from state k to state ℓ, π_k the stationary probability on state k and $(i,j,\ldots,j_{n-1},j)$ is a path of states such that the denominator is positive. Unfortunately, most models encountered in practice do not satisfy the Kolmogorov property. The quotient in the right hand side of (1.1) will then depend on the chosen path $(i,j_1,\cdots,j_{n-1},j)$. The idea of the bounding method discussed in [3],[7] and in this paper, is that the supremum over all paths from i to j gives an upperbound for the weighted stationary probability $\frac{\pi_j}{\pi_i}$. In [3] we point out by a counterexample that this supremum does not always give an upperbound. We need the following property on path reversibility, "the numerator and denominator of the quotient in (1.1) are

both zero or both positive for any path $(i,j_1,\cdots, j_{n-1},j)$".
Under the path reversibility property we show in section 2 that indeed the bounding method is valid if we take the supremum over all paths. This result is easily shown when using taboo probabilities. However, the bound is in most cases trivial. Indeed, if a cycle of states with a quotient unequal to one (see (2.4)) can be reached from state i then the upperbound is equal to infinity. In case such a cycle does not exist or cannot be reached from i the expression in the right hand side of (1.1) is again independent of the path and relation (1.1) holds.
From this analysis it becomes clear that we should take the maximum over all paths without cycles. Although, the bound indeed remains valid with the maximum over the simple paths it is not so easily proven. We need a special representation of the stationary distribution called the tree-form solution. In section 3 we review this representation and its application to obtain the desired upperbound.
In section 4 a reversible Markov chain is constructed (construction 4.1) for which the weighted stationary probabilities are equal to the upperbound.In [3] and [11] the method is applied to overflow models. The numerical evaluation there indicates that the bound is not very accurate. However, together with a lower bound (obtained by taking the minimum over all simple paths) a two sided inequality for the weighted stationary probabilities is available.
It is not surprising that the upper and lower bounds are mostly not tight. The bounding Markov chains are reversible and hence we know that their stationary distribution are insensitive with respect to the life time (for queueing models the service times) distributions. So the bounded stationary probabilities are varying with different distributions, while the bounds remain the same (if the mean life times do not change). For a class of nonexponential life time distributions the bounds remain valid. This class contains the phase-type distributions which can be generated by a reversible Markov chain (see [7]). Whether they remain valid for all distributions is an open problem. In most cases bounds on the stationary probabilities are desired instead of bounds on the weighted ones. In the constructions 4.2 and 4.3 we derive stochastic inequalities for the unweighted stationary distributions. For doing so we assume a partial ordering on the state space in 4.3 and we introduce one in 4.2. Under a special assumption on the transition rates (a generalization of the skip free property to the right together with path reversibility for a set of pairs of states) we construct reversible Markov chains which give stochastic upperbounds (theorems 4.2.3 and 4.3.1). These bounds are obtained by taking the maximum over all paths having minimal lengths and therefore may expected to be more accurate.
Finally in section 5 we give examples illustrating the constructions and bounds.

2. PATH REVERSIBLE MARKOV CHAINS

Let $\{X(t) : t \geq 0\}$ be an irreducible Markov chain on a finite statespace E with infinitesimal generator $Q = (q_{ij})_{i,j \in E}$. We assume that there are no instantaneous transitions, thus $q_{ij} < \infty$ for all $i,j \in E$, and therefore the chain is uniformizable (cf. [5]). By uniformization we obtain an irreducible Markov chain $\{X(n) : n = 0,1,\ldots\}$ on E with matrix of transition probabilities $P = (p_{ij})_{i,j \in E}$ which is determined by

$$P = I + \frac{1}{\gamma} Q ,$$

where I is the identical matrix on E×E and γ sufficiently large (cf. [5],[8]). It is wellknown that the stationary distributions of the two chains exist on E and that they are the same stationary distribution. We write $\pi = (\pi_i)_{i \in E}$ for this distribution.

For $i,j \in E$, $i \neq j$ we can interpret the ratio $\frac{\pi_j}{\pi_i}$ as the expected number of visits of the discrete-time chain state j between two visits to state i. If we denote the taboo transition probabilities with ${}_kp_{ij}^{(n)}$, $i,j,k \in E$, $n = 1,2,\ldots$, that is

$${}_kp_{ij}^{(n)} = P(X(n) = j,\ X(m) \neq k,\ m = 1,\ldots,n-1 \,|\, X(0) = i) ,$$

then we can translate this interpretation to

$$\frac{\pi_j}{\pi_i} = \sum_{n=1}^{\infty} {}_ip_{ij}^{(n)} \tag{2.1}$$

(cf. also [1],[2]).
In this section we shall derive an upperbound of the ratio $\frac{\pi_j}{\pi_i}$ in terms of the transition probabilities. To assure this upperbound exists we introduce a reversibility property of the transitions in the following way.

<u>DEFINITION 2.1.</u>
Let $i,j \in E$, $i \neq j$.

(i) A path from i to j is a sequence of states $(i = j_0, j_1, \ldots, j_{n-1}, j_n = j)$ with $j_1, j_2, \ldots, j_{n-1} \neq i$ and with $p_{j_\ell j_{\ell+1}} > 0$, $\ell = 0,1,\ldots,n-1$.

(ii) State i is path reversible related to j if any path $(i = j_0, j_1, \ldots, j_{n-1}, j_n = j)$ from i to j satisfies $p_{j_{\ell+1} j_\ell} > 0$, $\ell = 0,1,\ldots,n-1$.

The following lemma expresses our first bounding result.

LEMMA 2.2.

If i is path reversible related to j ($i \neq j$), then

$$(2.2)\qquad \frac{\pi_j}{\pi_i} \le \sup \frac{p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j}}{p_{j_1i}p_{j_2j_1}\cdots p_{jj_{n-1}}},$$

where the supremum is taken over all paths $(i,j_1,j_2,\ldots,j_{n-1},j)$ $n \in \mathbb{N}$, from i to j.

PROOF.

Let σ be the right-hand side of (2.2). Then for any $n \in \mathbb{N}$

$$ {}_iP_{ij}^{(n)} = \sum_{j_1,j_2,\ldots,j_{n-1}\neq i} p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j}$$

$$\le \sigma \sum_{j_1,j_2,\ldots,j_{n-1}\neq i} p_{j_1i}p_{j_2j_1}\cdots p_{jj_{n-1}} =: \sigma f_{ji}^{(n)},$$

where $f_{ji}^{(n)}$ denotes the probability of being at time n for the first time in state i when starting in j. This inequality gives with (2.1)

$$\frac{\pi_j}{\pi_i} \le \sigma \sum_{n=1}^{\infty} f_{ji}^{(n)} = \sigma . \qquad \square$$

Having the bounding result (2.2) for the discrete time Markov chain we can easily extend it to the continuous time chain.

COROLLARY 2.3.

If i is path reversible related to j ($i \neq j$), then

$$(2.3)\qquad \frac{\pi_j}{\pi_i} \le \sup \frac{q_{ij_1}q_{j_1j_2}\cdots q_{j_{n-1}j}}{q_{j_1i}q_{j_2j_1}\cdots q_{jj_{n-1}}},$$

where the supremum is taken over all paths $(i,j_1,j_2,\ldots,j_{n-1},j)$, $n \in \mathbb{N}$, from i to j.

PROOF.

Any factor $p_{j_\ell j_\ell}$ in the right-hand side of (2.2) occurs both in the numerator and in the denominator and therefore may be omitted. Hence the right-hand sides of (2.2) and (2.3) are equal because for $j_\ell \neq j_{\ell+1}$

$$p_{j_\ell j_{\ell+1}} = \frac{1}{\gamma} q_{j_\ell j_{\ell+1}} \quad . \qquad \square$$

In the rest of this section we concentrate on the (original) continuous time Markov chain. In the bound of (2.3) we allow the paths to contain cycles. If $(j_\ell, j_{\ell+1}, \ldots, j_{m-1}, j_m = j_\ell)$ is a cycle which occurs in some path from i to j and for which

$$(2.4) \qquad \frac{q_{j_\ell j_{\ell+1}} q_{j_{\ell+1} j_{\ell+2}} \cdots q_{j_{m-1} j_\ell}}{q_{j_{\ell+1} j_\ell} q_{j_{\ell+2} j_{\ell+1}} \cdots q_{j_\ell} q_{j_{m-1}}} \neq 1 \quad ,$$

then the upperbound given in (2.3) becomes infinitely large by inserting an infinite repetition of this cycle such that the ratio of (2.4) is larger than one. Hence, generally the bounding results (2.2) and (2.3) are trivial. They will be not trivial if we may take in (2.2) and (2.3) the supremum over all paths without cycles.

Let us call paths without cycles simple paths. In [3] the bound (2.3) with the supremum taken only over the simple paths is shown to be valid. The derivation of this result uses a representation of the stationary distribution of the chain which is called the tree-form solution (cf. [9]).

3. THE TREE-FORM SOLUTION

Associated with the Markov chain $\{X(t) : t \geq 0\}$ of section 2 is a directed graph $G = (E,A)$ with vertex set the statespace E of the chain and arcs $(i,j) \in A$ $i \neq j$ if and only if the transition rate q_{ij} is positive. A rooted tree $T = (E,A_T)$ is a subgraph of G without cycles consisting of the same vertices as G and with outdegree of each vertex at most one. The vertices with outdegree zero in T are called the roots of the tree. We denote the set of all roots of T with ρ_T (note that $\rho_T \neq \phi$). Associated with T is a treefunction $f_T = E \to E \cup \{0\}$ defined by $f_T(i) = j$ if $(i,j) \in A_T$ and $f_T(i) = 0$ if $i \in \rho_T$.

DEFINITION 3.1

The tree-form vector $\sigma = (\sigma_i)_{i\in E}$ is defined by

$$\sigma_i = \sum_{T:\rho_T = \{i\}} \prod_{j\neq i} q_{jf_T(j)} \quad (3.1)$$

In [9] it is shown that σ is a solution of $xQ = 0$. Hence we obtain the tree-form solution of the stationary distribution of the chain by normalizing σ,

$$\pi_i = \frac{\sigma_i}{\sum_{i\in E} \sigma_i} \quad , \quad i \in E . \quad (3.2)$$

Using the tree-form representation of the stationary distribution we derived in [3] that indeed the right-hand side of (2.3) restricted to the simple paths gives an upperbound.

We denote with W_{ij} the set of all simple paths from i to j in the graph G. If i is path reversible related to j and $w = (i = j_0, j_1, \cdots, j_{n-1}, j_n = j) \in W_{ij}$ then

$$h_{ij}(w) = \frac{q_{ij_1}\, q_{j_1j_2} \cdots q_{j_{n-1}j}}{q_{j_1i}\, q_{j_2j_1} \cdots q_{jj_{n-1}}} \quad (3.3)$$

is properly defined. In [3] it is shown that

$$\frac{\pi_j}{\pi_i} \leq \sup_{w\in W_{ij}} h_{ij}(w) . \quad (3.4)$$

4. STOCHASTIC BOUNDS THROUGH REVERSIBLE MARKOV CHAINS

Again we start with the irreducible Markov chain $\{X(t) : t \geq 0\}$ of section 2. We fix some state $i_0 \in E$ as reference state. We shall construct several reversible Markov chains which provide stochastic upperbounds for the stationary distribution. Examples of these constructions are given in section 5.

CONSTRUCTION 4.1

Assume i_0 is path reversible related to all other states $j \in E$. Let $h_j, j \in E$, be the right-hand side in (3.4), then we have

(4.1) $$\pi_j \le \pi_{i_0} h_j \ , \ j \in E \ .$$

We define a Markov chain $\{X^{(1)}(t) : t \ge 0\}$ on E with transition rates $(q_{ij}^{(1)})_{i,j\in E}$ by

(4.2) $$q_{ij} = q_{ji} = 0 \Leftrightarrow q_{ij}^{(1)} = q_{ji}^{(1)} = 0$$

and

(4.3) $$h_i q_{ij}^{(1)} = h_j \, q_{ji}^{(1)} \ ,$$

for all $i,j \in E$, $i \neq j$.

We read in this construction that both $q_{ij}^{(1)}$ and $q_{ji}^{(1)}$ are positive if at least one of q_{ij} and q_{ji} is positive; the new rates may be chosen freely, as long as they satisfy (4.3). From (4.3) we see that the new chain is reversible by applying Kolmogorov's criterium (cf. [6]). Furthermore we can derive from (4.3) that the stationary distribution $\pi^{(1)}$ satisfies $\pi_j^{(1)} = \pi_{i_0}^{(1)} h_j$ for $j \in E$. Together with (4.1) that means

(4.4) $$\frac{\pi_j}{\pi_{i_0}} \le \frac{\pi_j^{(1)}}{\pi_{i_0}^{(1)}} \ , \ j \in E \ .$$

<u>CONSTRUCTION 4.2</u>

Let ℓ_w be the length of path w in the graph G, by which we mean the number of arcs in w. Then

$$\ell_j^* := \min_{w \in W_{i_0 j}} \ell_w \ , \ j \in E$$

is the minimal length of simple paths from the reference state i_0 to j. We do not assume path reversibility in the graph but we do assume the chain $\{X(t) : t \ge 0\}$ satisfies the following conditions.

(4.5) $$q_{ij} > 0 \Rightarrow \ell_j^* - \ell_i^* \in \{\cdots,-2,-1,1\}$$

and

$$\left.\begin{array}{l} q_{ij} > 0 \\ \ell_j^* - \ell_i^* = 1 \end{array}\right\} \Rightarrow q_{ji} > 0 \ , \qquad (4.6)$$

for all $i,j \in E$, $i \neq j$.

Remark that if $w = (i_0, i_1, \ldots, i_n = j) \in W_{i_0 j}$ is a path with minimal length, then also any beginning of w, $w_k = (i_0, i_1, \ldots, i_k) \in W_{i_0 i_k}$, is a path with minimal length. With this observation we endow the statespace E with a partial ordering as follows.

DEFINITION 4.2.1

Let $i,j \in E$, $i \neq j$. Then $i \leq j$ iff there exists $w = (i = i_1, i_2, \ldots, i_n = j) \in W_{ij}$ such that $\ell_{i_{k+1}}^* - \ell_{i_k}^* = 1$ for all $k=1,2,\ldots,n-1$, or $w = (j = i_1, i_2, \ldots, i_n = i) \in W_{ij}$ such that $\ell_{i_{k+1}}^* - \ell_{i_k}^* \leq -1$ for all $k=1,2,\ldots,n-1$.

By this definition "$\leq$" is indeed a partial ordering on E, with $i_0 \leq j$ for all $j \in E$. The condition (4.6) says that if it is possible to do one transition "upwards", then also the "downwards" transition between the same pair of states is possible. If the two conditions (4.5) and (4.6) are satisfied we say that the graph G is path reversible related from below with respect to i_0. In that case the following function is finite and positive for $j \in E$,

$$h_j^* := \sup_{w \in W_{i_0 j},\ \ell_w = \ell_j^*} h_{i_0 j}(w) \qquad (4.7)$$

(for $h_{i_0 j}(w)$ see (3.3)). The difference with (3.4) is the set of paths over which we take the supremum.

Now we construct a Markov chain $\{X^{(2)}(t) : t \geq 0\}$ on E by defining its transition rates $(q_{ij}^{(2)})_{i,j \in E}$ as follows.

$$q_{ij} q_{ji} = 0 \Leftrightarrow q_{ij}^{(2)} = q_{ji}^{(2)} = 0 \ , \qquad (4.8)$$

$$\left.\begin{array}{l} q_{ij} q_{ji} > 0 \\ \ell_j^* - \ell_i^* = 1 \end{array}\right\} \Rightarrow q_{ji}^{(2)} = q_{ji} \qquad (4.9)$$

and

(4.10) $$h_i^* q_{ij}^{(2)} = h_j^* q_{ji}^{(2)}$$

for all $i,j \in E$, $i \neq j$.

We read in this construction that both $q_{ij}^{(2)}$ and $q_{ji}^{(2)}$ are zero if at most one of q_{ij} and q_{ji} is positive. The new positive downward transition rates are the same as the original ones whereafter the new positive upward transition rates follow from (4.10). It is easy to see from (4.10) that the new chain $\{X^{(2)}(t) : t \geq 0\}$ is reversible. Let $\pi^{(2)}$ be its stationary distribution on E. In order to formulate a comparison result between π and $\pi^{(2)}$ we introduce the usual stochastic ordering (cf. [5], [10]).

DEFINITION 4.2.2

(i) $I \subset E$ is an increasing set if $i \in I$, $i \leq j$ implies $j \in I$.

(ii) Two probability measures $p^{(1)}$ and $p^{(2)}$ on E are stochastically ordered as $p^{(1)} \leq p^{(2)}$ if $p^{(1)}(I) \leq p^{(2)}(I)$ for all increasing sets I.

(iii) A Markov chain $\{Y(t) : t \geq 0\}$ on E is stochastically monotone if $P(Y(t) \in I | Y(0) = i) \leq P(Y(t) \in I | Y(0) = j)$ for all $t \geq 0$, increasing sets I and $i \leq j$.

Now we claim to have the following result.

THEOREM 4.2.3

If $\{X(t) : t \geq 0\}$ or $\{X^{(2)}(t) : t \geq 0\}$ is stochastically monotone, then

(4.11) $$\pi \leq \pi^{(2)}.$$

PROOF.

To show (4.11) the comparison techniques of Markov chains as described in [10], chapter 4, can be applied. In [7] this is carried out in extenso. □

CONSTRUCTION 4.3

Assume "≤" is a partial ordering defined on the statespace E and the chain satisfies the following conditions.

(4.12) $$q_{ij} > 0 \Rightarrow \ell_j^* - \ell_i^* = 1 \quad \text{or} \quad j \leq i$$

and

$$(4.13)\qquad \left.\begin{array}{l} q_{ij} > 0 \\ \ell_j^* - \ell_i^* = 1 \end{array}\right\} \Rightarrow q_{ji} > 0 \quad ,$$

for all $i,j \in E$, $i \neq j$.

Condition (4.13) is the same as (4.6). Contrary to definition 4.2.1 it is now possible to have $\ell_j^* - \ell_i^* = 1$ and $j \le i$. The two conditions (4.12) and (4.13) imply that the function h_j^* of (4.7) is finite and positive for all $j \in E$. The construction of the chain $\{X^{(3)}(t) : t \ge 0\}$ on E with transition rates $(q_{ij}^{(3)})_{i,j\in E}$ goes along similar lines as in construction 4.2.

$$(4.14)\qquad q_{ij}q_{ji} = 0 \Leftrightarrow q_{ij}^{(3)} = q_{ji}^{(3)} = 0 \quad ,$$

$$(4.15)\qquad \left.\begin{array}{l} q_{ij}q_{ji} > 0 \\ i \le j \end{array}\right\} \Rightarrow q_{ji}^{(3)} = q_{ji}$$

and

$$(4.16)\qquad h_i^* q_{ij}^{(3)} = h_j^* q_{ji}^{(3)}$$

for all $i,j \in E$, $i \neq j$.

The new chain $\{X^{(3)}(t) : t \ge 0\}$ is reversible (follows easily from (4.16)). Similar to theorem 4.2.3 we formulate a comparison result for the stationary distributions.

THEOREM 4.3.1.

If $\{X(t) : t \ge 0\}$ or $\{X^{(3)}(t) : t \ge 0\}$ is stochastically monotone, then

$$(4.17)\qquad \pi \le \pi^{(3)} \; . \qquad \square$$

REMARK 4.4.

If we read the constructions 4.2 and 4.3 carefully, we note that we delete any original transition whenever the backward transition does not exists (cf. (4.8) and (4.14)). The remaining transitions receive rates such that the new chains are reversible. It is possible to make the transition rates positive between any pair of states and to preserve the reversibility and the stationary distribution by taking care that the Kolmogorov criterium also holds with the two added transition rates (cf. [6]). Hence, if we put any deleted transition with its original rate back to the constructed reversible chain, we must add its (originally nonexisting) backward transition with an appropriate rate so that the reversibility is not destroyed and the stationary distribution is not changed. In this way we obtain reversible versions of $\{X^{(2)}(t) : t \geq 0\}$ (resp. $\{X^{(3)}(t) : t \geq 0\}$), all with the same stationary distribution $\pi^{(2)}(\pi^{(3)})$. That means that the conditions of the theorems 4.2.3 and 4.3.1 can be relaxed somehow.

5. APPLICATIONS

In this section we illustrate the constructions of this paper by counting-examples. The inequalities (3.4), (4.11) and (4.17) have also been applied to natural models. In [3] and [11] overflow models are analyzed and other models will be studied in [7].
In [12] finite networks of queues are modelled by jobmark processes. A jobmark process $\{X(t) : t \geq 0\}$ generally describes a system of finitely many (say K) jobmarks each of which is repeatively activated and dying out. The states of this process are K-tuples $i = (i_1, i_2, \ldots, i_K)$ with $i_k \in \{0,1\}$, $k = 1,2,\ldots,K$. The k-th component indicates whether the k-th jobmark is active ($i_k = 1$) or dead ($i_k = 0$). We assume the process $\{X(t) : t \geq 0\}$ satisfies the following restrictions.

(5.1) Only one jobmark can change at a time.

(5.2) The lifetime of an active jobmark is exponentially distributed with unit mean.

(5.3) Activation of a dead jobmark happens according to a Poisson process with an intensity that may depend on the state i of the system.

(5.4) All lifetimes of active jobmarks and activations of dead jobmarks are independent.

(5.5) The process $\{X(t) : t \geq 0\}$ is irreducible.

With these assumptions the jobmark process is an irreducible Markov chain on $E = \{0,1\}^K$ with transition rates $(q_{ij})_{i,j \in E}$ which satisfy

(5.6) $q_{ij} = 0$ if i and j differ in more than one component, (5.7) $$q_{i,i-e_k} = 1 \text{ if } i_k = 1 ,$$

where e_k is the state with the k-th jobmark as only living jobmark, $i - e_k$ is the usual vector subtraction. Jobmark processes are described more detailed in [12] and are closely related to the spatial processes of [6], chapter 9. To show how the constructions of section 4 work we give two simple examples with three jobmarks.

5.1. We denote with $|i|$ the number of active jobmarks in state i ($|i| = i_1 + i_2 + i_3$). Suppose jobmark 1 is activated if $|i| \in \{0,1\}$ (and $i_1=0$), jobmark 2 if $|i| \in \{1,2\}$ (and $i_2=0$) and jobmark 3 only in state i=(010). In the associated graph of this model the empty state i_0=(000) is path reversible related to all other states. Also, the transition rates satisfy the conditions (4.5) and (4.6) of construction 4.2. The partial ordering of definition 4.2.1 is the following total ordering,

(5.8) $$(000) \leq (100) \leq (110) \leq (010) \leq (011) \leq (001) \leq (101) \leq (111).$$

If the statespace E is endowed with the vector ordering, then conditions (4,12) and (4.13) of construction 4.3 are fulfilled.

We label the successive states of ordering (5.8) from 0 up to 7. Then we obtain for n=1,2,...,7

$$h_n = h_n^* = \prod_{m=0}^{n-1} \frac{q_{m,m+1}}{q_{m+1,m}} .$$

This means that the three constructed reversible chains are equivalent in the sense $\pi^{(1)} = \pi^{(2)} = \pi^{(3)}$. For these chains the transition rates between two successive states of ordering (5.8) are the original ones. Furthermore, $\{X^{(2)}(t) : t \geq 0\}$ is stochastically monotone because it is a birth death process on {0,1,...,7} (cf. [5]). It is possible to introduce a stochastically monotone reversible version of $\{X^{(3)}(t) : t \geq 0\}$ (cf. remark 4.4) if for all $i \in E$ with $i_k = i_\ell = 0$

(5.9) $$\frac{h^*_{i+e_k}}{h^*_i} \leq \frac{h^*_{i+e_k+e_\ell}}{h^*_{i+e_\ell}}$$

(cf. [7]).

5.2. Suppose jobmark 1 is activated if $|i| = 0$ or $i_2 = 1$ (and $i_1 = 0$), jobmark 2 if $i_1 = 1$ (and $i_2 = 0$) and jobmark 3 if $|i| \in \{0,2\}$ (and $i_3 = 0$). The reference state is again $i_0 = (000)$. In the associated graph i_0 is not path reversible related to (001), hence construction 4.1 fails. The conditions of the constructions 4.2 and 4.3 are fulfilled.
Because for any $j \neq i_0$ there is only one path $w \in W_{i_0 j}$ with minimal length ($\ell_w = \ell_j^*$) the reversible chains $\{X^{(2)}(t) : t \geq 0\}$ and $\{X^{(3)}(t) : t \geq 0\}$ are similar and obtained from the original chain by deleting the rates $q_{i+e_k,i}$ if $q_{i,i+e_k} = 0$. Again (5.9) is sufficient to introduce a stochastically monotone reversible version of $\{X^{(3)}(t) : t \geq 0\}$. In that case $\pi \leq \pi^{(3)}$. With the partial ordering of definition 4.2.2 it can be shown that $\{X^{(2)}(t) : t \geq 0\}$ is not stochastically monotone.

REFERENCES

[1] Chung, Kai Lai, Markov chains with stationary transition probabilities, Berlin, Springer (1960).

[2] Feller, W., An introduction to probability theory and its applications I, New York, Wiley, 1968.

[3] Hordijk, A. & Ridder, A., Insensitive bounds for the stationary distribution of nonreversible Markov chains, to appear in J.Appl.Prob. (1988).

[4] Keilson, J., Markov chain models - rarity and exponentiality, New York, Springer (1979).

[5] Keilson, J. & Kester, A., Monotone matrices and monotone Markov processes, Stoch. Proc. Appl. 5, 231-241 (1977).

[6] Kelly, F., Reversibility and stochastic networks, New York, Wiley (1979).

[7] Ridder, A., Stochastic inequalities for queues, dissertation, Leiden (1987).

[8] Ross, S.M., Stochastic processes, New York, Wiley (1983).

[9] Schoute, A.L., Modelling of virtual memory computer systems with multiple classes of processes, dissertation, Groningen (1978).

[10] Stoyan, D., Comparison methods for queues and other stochastic models, New York, Wiley (1983).

[11] Hordijk, A. & Ridder, A., Stochastic inequalities for an overflow model, to appear in J.Appl.Prob. (1987).

[12] Hordijk, A. & Van Dijk, N., Networks of queues, Modelling and Performance Evaluation Methodology (eds. F. Baccelli, G. Fayolle), Lecture Notes in Control and Information Sciences 60, Berlin, Springer, 151-205 (1983).