Risk Aversion under Preference Uncertainty

Roman Kraeussl André Lucas Arjen Siegmann*

VU University Amsterdam

This version: July 1, 2011

Abstract

We show that if an agent is uncertain about the precise form of his utility function, his actual relative risk aversion may depend on wealth even if he knows his utility function lies in the class of constant relative risk aversion (CRRA) utility functions. We illustrate the consequences of this result for optimal asset allocation: poor agents that are uncertain about their risk aversion parameter invest less in risky assets than wealthy investors with identical risk aversion uncertainty.

JEL classification: D81, D84, G11

Keywords: risk aversion; preference uncertainty; risk-taking; optimal asset allocation.

1. Introduction

Under power utility, the optimal asset mix in the standard lifetime consumption and investment setup does not depend on wealth. This result dates back to the pioneering work of Merton (1971). It follows directly from the fact that power utility has constant relative risk aversion (CRRA). The result has been a source of ongoing controversy between academics and practitioners, in particular relating to the issue of 'time-diversification'. Time-diversification builds on the intuitive argument that repeated risky bets with positive expected return are

*Corresponding author, VU University Amsterdam, FEWEB/Fin, De Boelelaan 1105, 1081 HV, The Netherlands, a.h.siegmann@vu.nl. All three authors are at the VU University. Lucas is also with the Duisenberg School of Finance and the Tinbergen Institute.

less risky and more desirable than their single-bet counterparts. The claimed reason for this is that the risk is diversified over time. Samuelson (1963) already showed that this argument is flawed if the decision makers risk aversion coefficient is constant in the level of wealth. Attempts to resolve the controversy typically introduce further complications in the decision-making framework, such as habit formation or a distinction between total and financial wealth, see for example Bodie et al. (2004), Polkovnichenko (2007), and Cocco et al. (2005).

In this paper, we show how uncertainty about the risk aversion coefficient directly leads to a wealth-dependent optimal asset allocation. We do so without leaving the CRRA power utility specification. The intuition is that the effective risk aversion facing a preference-uncertain agent is not a simple average over the possible realizations of risk aversion, but rather a weighted average. The key point is that the weights depend on wealth, which stems from the dependence of marginal utility on both the wealth level and the risk aversion coefficient. For low levels of wealth in a power utility setting, marginal utility is steeply increasing in risk aversion. As a result, preference uncertainty leads to a high weight for high-risk-aversion utilities and, therefore, to a lower allocation to the risky asset. For high levels of wealth and power utility, marginal utility decreases in risk aversion and utilities corresponding to low risk aversion levels dominate the asset allocation problem. This results in a higher allocation to the risky asset. In this paper we formalize the intuition above by formally proving wealth dependence for general utility functions and deriving the consequences for optimal asset allocation in the case of power utility.

The wealth-dependence resulting from preference uncertainty offers a new explanation for an empirical stylized fact. We know that households display behavior consistent with both CRRA and decreasing relative risk aversion (DRRA) preferences, see Levy (1994) and Ogaki and Zhang (2001). DRRA is equivalent to observing a higher investment in risky assets for higher wealth levels. This empirical evidence is not consistent with households having power utility preferences with a fixed risk aversion parameter. The explanation following from our current analysis is that households may be uncertain with respect to their risk aversion parameter when evaluating future investment outcomes. This simple form of uncertainty about preferences leads to a positive relationship between wealth levels and risky investments.

We model preference uncertainty by considering an agent who does not know his preferences over different (risky) final wealth levels when making his investment decision at time 0. We view this uncertainty as originating from the agent considering different future 'selves' and assigning probabilities to each of these. Each of these future selves has a different risk aversion parameter, see for example Brunnermeier and Parker (2005) in the context of biased beliefs.

The assumption that individuals may face uncertainty about their future preference structure and in particular about the exact level of their risk aversion coefficient is supported by numerous empirical studies. Andersen et al. (2008) find sizable within-subject differences in elicited relative risk aversion. Weber and Milliman (1997) find that agents preferences change between the stages of thinking about the choice and actually making the choice. Finally, Fischer et al. (2000) argue that preference uncertainty may arise out of unfamiliarity with prospects that have multiple attributes and the need for learning about these prospects.

An alternative interpretation of our results relates to group decision-making processes. Within a group, different decision makers can have dissimilar preferences, while the group as a whole has to reach a single decision. A typical example is provided by an investment committee of a pension fund or of a university endowment fund. Here, each member or stakeholder can have a different risk appetite, while the fund in the end has to settle on a distinct asset allocation. A solution based on alternating offers, such as in Rubinstein (1982), or an equivalent Nash bargaining game then gives rise to a weighted average of the expected utilities of the different committee members that strongly resembles the objective function in our preference uncertainty setting. Our results underline that the outcome of a group decision-making process is not just the decision taken by the average committee member. Rather, the risk appetite of the group may be wealth dependent even if the risk appetite of every single committee member is not.

Assuming uncertainty about risk aversion can also be considered as a type of background risk, see for example Guiso and Paiella (2008) and Heaton and Lucas (2000). Traditionally, background risk typically affects the economic environment of the household, for instance, in the form of income and liquidity risk. In our current setting of preference uncertainty, the background risk relates to the uncertainty which of the possible future selves will evaluate the risky outcomes of the investment decision.

Our paper is also related to the framework of decision-making under ambiguity aversion by Klibanoff, Marinacci, and Mukerji (2005, 2009). They study the impact of uncertainty about the probabilities of risky outcomes. We take the probabilities of the risky outcomes as given and instead study the impact of uncertainty about the preference structure itself.

The existing literature on preference uncertainty is scant and mostly concerned with the deterministic case, where preference uncertainty is the stochastic element in an otherwise deterministic setting of resource valuation, see for example Li and Mattsson (1995), Akter et al. (2008), and Van Kooten et al. (2001).

The remainder of this paper is set up as follows. Section 2 formalizes our model for preference uncertainty and defines a risk aversion measure under uncertain preferences. Section 3 presents the consequences of preference uncertainty for optimal asset allocation in a power utility context. Section 4 concludes.

2. The model

Consider an agent with investment horizon T and uncertain final wealth W_T. The distribution of W_T is denoted as $F(W_T)$. The agent's utility function, $U(w, \gamma)$, is indexed by a parameter γ. For example, U can be a standard power utility function $U(W, \gamma) = (1-\gamma)^{-1}W^{1-\gamma}$, with γ the coefficient of relative risk aversion. We assume there is uncertainty about the precise value of γ. The uncertainty about γ is summarized by the distribution function $G(\gamma)$.

We assume the agent maximizes the following objective function

$$V(F, G) = \int v \left(\int U(w, \gamma) \mathrm{d}F(w) \right) \mathrm{d}G(\gamma), \tag{1}$$

where v is a strictly increasing concave function. The function v captures the agent's aversion to preference uncertainty in a similar way as the ambiguity aversion function of Klibanoff, Marinacci, and Mukerji (2005, 2009). The more curved the function v, the higher the agent's aversion to preference uncertainty.

Equation (1) can be viewed as representing the preferences of an agent who does not know his preferences over different consumption bundles at time 0, while having to make investment decisions that impact future pay-offs. The uncertainty about his risk aversion parameter,γ, is resolved at some later date when consumption actually takes place.

The uncertainty over γ can be seen as an agent having different future 'selves', each having a different risk aversion parameter. The different risk aversion parameters represent the possible states of mind in the future, see for example Bénabou and Tirole (2002) and Brunnermeier and Parker (2005).

Another interpretation is that multiple stakeholders with different preferences have to reach a single investment decision. This occurs for example in the context of pension funds or university endowment funds.

Alternatively, the uncertainty about γ can be considered as a type of background risk, as in Guiso and Paiella (2008) and Heaton and Lucas (2000). It enters the model exogenously and cannot be hedged completely. Usually, background risk enters the objective through F only. Here, by contrast, the background risk enters through the separate distribution function G, where G does not affect the distribution of final wealth. Instead, G operates on the perception of final wealth through the utility function and through the risk-return trade-off over all possible final wealth levels.

The objective function in (1) naturally embeds the expected utility case for fixed γ. To see this, note that if δ_g is the Dirac function that jumps from 0 to

1 at $g \in \mathbb{R}$, we obtain

$$V(F, \delta_g) = v\left(\int U(w, g)\mathrm{d}F(w)\right), \tag{2}$$

such that maximizing $V(F, \delta_g)$ is the same as maximizing expected utility. Also note that if $v(\cdot)$ is linear, the problem is consistent with expected utility, where the expectation is taken with respect of both risky wealth (w) and the 'risky' preference parameter (g).

The first and second order partial derivatives of U with respect to wealth are denoted as U' and U'', respectively. We assume $U' > 0$, and $U'' < 0$. We also introduce $\bar{U} = \bar{U}(\gamma) = \int U(w, \gamma)\mathrm{d}F(w)$ as a short-hand notation for expected utility under the fixed preference γ, with first and second order derivatives (with respect to γ) denoted as $\bar{U}'$ and $\bar{U}''$, respectively. Finally, v' and v'' denote the first and second order derivatives of v with respect to its argument $\bar{U}$, and we assume $v' > 0$ and $v'' < 0$.

A key concept to assess the effect of preference uncertainty on optimal asset allocation is the Arrow-Pratt coefficient of absolute risk aversion (ARA). A textbook result is that for standard utility functions, the ARA coefficient can be obtained by computing the negative second order derivative of the certainty equivalent for a small Bernoulli gamble, i.e., $\mathrm{ARA} = -\partial^2 c(e)/\partial e^2|_{e=0}$, where

$$U(c(e)) = 0.5U(W_0 + e) + 0.5U(W_0 - e),$$

see also the Appendix. Using this definition for absolute risk aversion, the following theorem gives our main result.

Theorem 2.1. *Define the risk aversion coefficients with respect to w and γ as $\mathrm{ARA}_w = -\partial^2 c(e)/\partial e^2|_{e=0}$ and $\mathrm{ARA}_\gamma = -\partial^2 g(e)/\partial e^2|_{e=0}$, where*

$$\begin{aligned} V(\delta_c, G) &= V(0.5\delta_{w-e} + 0.5\delta_{w+e}, G), && (3) \\ V(F, \delta_g) &= V(F, 0.5\delta_{\gamma-e} + 0.5\delta_{\gamma+e}), && (4) \end{aligned}$$

for fixed w and γ. We have

$$\mathrm{ARA}_w = -\int \frac{U''(w, \tilde{\gamma})}{U'(w, \tilde{\gamma})}\ \mathrm{d}\bar{G}(\tilde{\gamma}, w), \tag{5}$$

where

$$\mathrm{d}\bar{G}(\tilde{\gamma}, w) = \frac{v'(U(w,\tilde{\gamma}))U'(w,\tilde{\gamma})\mathrm{d}G(\tilde{\gamma})}{\int v'(U(w,\tilde{\gamma}))U'(w,\tilde{\gamma})\mathrm{d}G(\tilde{\gamma})}. \tag{6}$$

Similarly,

$$\mathrm{ARA}_{\gamma} = -\frac{v''(\bar{U})}{v'(\bar{U})}\,\bar{U}' - \frac{\bar{U}''}{\bar{U}'}, \tag{7}$$

where $\bar{U} = \bar{U}(\gamma) = \int U(w,\gamma)\mathrm{d}F(w)$.

The proof of Theorem 2.1 can be found in the Appendix.

Equation (5) shows that under preference uncertainty the risk aversion coefficient with respect to wealth, ARA_w, is the expected value of the standard risk aversion coefficient for known γ. The expectation, however, is not taken with respect to the distribution G of preference uncertainty, but rather with respect to $\bar{G}$, as defined in Equation (6). The denominator in (6) is the integrating constant to ensure that $\bar{G}$ is a distribution function. The distribution $\bar{G}$ assigns more weight to those values of γ that have a high marginal utility U' for the current level of wealth w and/or a high marginal valuation v' of expected utility preference. In this way, the risk aversion coefficient becomes wealth dependent, even if the risk aversion coefficient of U itself does not depend on wealth.

For the case of power utility we give a clear illustration of the resulting wealth-dependence in the next section. Interestingly, the transform from G to $\bar{G}$ resembles the transform from actual to risk neutral probabilities via a pricing kernel, see for example Cochrane (2001). In this case, however, the transform is not applied to wealth uncertainty, but to preference uncertainty.

The risk aversion coefficient for preference uncertainty (ARA_{γ}) is composed of two terms. The first term of (7) reflects the curvature of v, which operates on expected utility. Clearly, the more curved v, the higher ARA_{γ}. The effect is multiplied by the derivative of expected utility with respect to γ. If expected utility has a low sensitivity to changes in γ, the curvature of v matters less. The second component of ARA_{γ} is the curvature of expected utility $\bar{U}$ with respect to γ (rather than w). Though the notation is similar to the standard notation for risk aversion, the expression for familiar utility specifications U is substantially different. For example, even for the case of power utility, no closed form expressions for $\bar{U}$ are readily available.

3. Asset allocation with CRRA utility

To illustrate Theorem 2.1, we now derive the optimal asset allocation for power utility and a two-point probability distribution of risk aversion. Consider an expected utility maximizing agent ($v(\bar{U}) = \bar{U}$), endowed with a power utility function

$$U(W_T, \gamma) = (1-\gamma)^{-1} W_T^{1-\gamma}, \tag{8}$$

where $\gamma = -W_T U''/U'$ denotes the relative risk aversion of the agent. The uncertainty about γ is such that it can take either a high value γ^H or a low value γ^L, with equal probability.

In the context of optimal asset allocation, the agent can invest in a risky and a risk-free asset with returns $r^f + r$ and r^f, respectively. The risky asset's excess return above the risk-free rate r has the probability distribution $F(r)$. If α denotes the fraction invested in the risky asset, end-of-period wealth W_T equals $W_T = W_0 \cdot (1 + r^f + \alpha \cdot r)$. Using Theorem 2.1, we obtain the relative risk aversion coefficient

$$\mathrm{RRA}_w = W \cdot \mathrm{ARA}_w = \frac{\gamma_L W^{\Delta\gamma} + \gamma_H}{W^{\Delta\gamma} + 1}, \tag{9}$$

where $\Delta\gamma = \gamma_H - \gamma_L > 0$. The RRA_w coefficient clearly depends on wealth, even though the RRA_w for fixed γ does not. Looking more closely at (9), we see that risk aversion monotonically decreases in W with an upper limit γ_H for small values of W, and a lower limit γ_L for high values of W. Put differently, the uncertainty in γ induces decreasing relative risk aversion (DRRA). Figure 1 illustrates the result.

The baseline case in Figure 1 is the setting without preference uncertainty: $\gamma^H = \gamma^L = 5$. We obtain the familiar result that the fraction invested in the

<< INSERT Figure 1 HERE >>

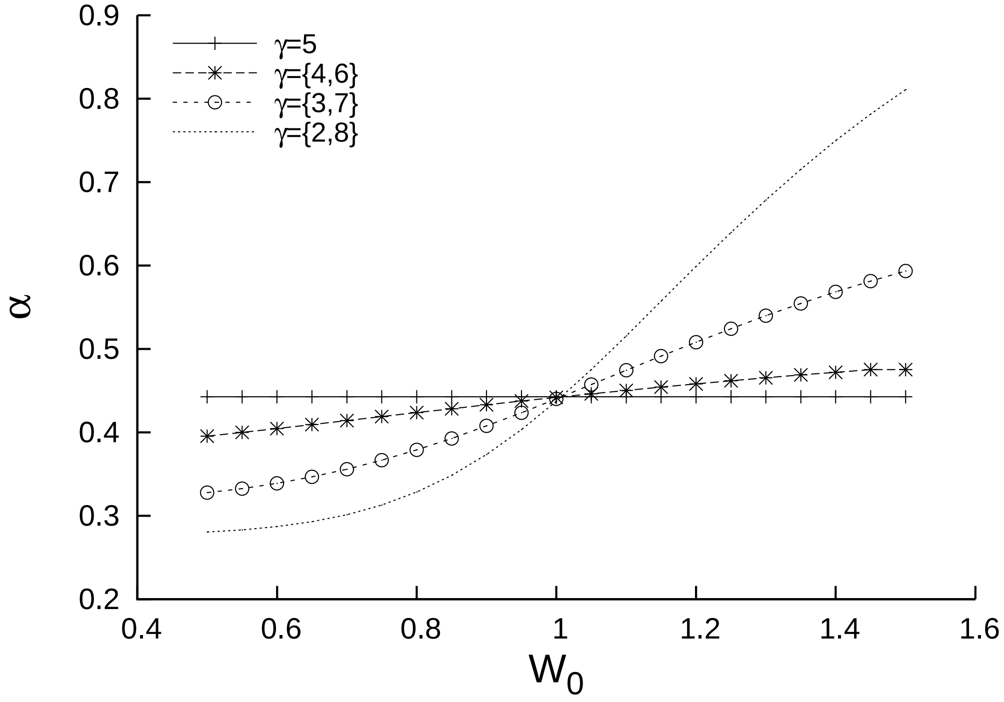

Figure 1: Optimal fraction in the risky asset

The figure shows the optimal fraction in the risky asset for uncertain γ, where γ takes a low or high value with equal probability as indicated in the legend. The horizontal line in the figure corresponds to $\gamma = 5$, which is the baseline case, the setting without preference uncertainty.

risky asset is constant in the initial level of wealth. If the uncertainty in γ is increased by a mean preserving spread, the pattern changes substantially. For high levels of wealth, the relative risk aversion coefficient in (9) is substantially lower than 5. This results in higher allocations to the risky asset. Ultimately, the allocation tends to that for γ^L. For low wealth levels, a similar result emerges. For low levels of wealth, the agent becomes more prudent, ultimately converging to the allocation for γ^H.

All curves in Figure 1 cross the point $(1, \alpha_{\bar{\gamma}})$, where $\alpha_{\bar{\gamma}}$ is the optimal asset allocation for the expected level of risk aversion $\bar{\gamma} = (\gamma^H + \gamma^L)/2$. Note that at

$W = 1$, $\bar{\gamma}$ is the risk aversion for $0.5U(1,\gamma^L)+0.5U(1,\gamma^H)$. This result indicates that under preference uncertainty, scaling of wealth starts to matter. This feature is shared with other utility functions without preference uncertainty, such as the exponential or constant absolute risk aversion (CARA) utility function.

The effect of preference uncertainty appears negligible if wealth at the horizon W_T is scaled by current wealth W_0, i.e., around the point $W = 1$ in Figure 1. However, this only holds in the static one-period model presented above. For the general multi-period context, wealth drifts from its initial starting value W_0 as time progresses. This re-introduces the changing asset allocations over wealth at later stages. As a result, our current setup produces succinctly different results from the familiar Merton-Samuelson multi-period result for CRRA utility functions without preference uncertainty, see Merton and Samuelson (1990). Our results offer a potential explanation for the observed decreasing relative risk aversion (DRRA) behavior in experimental studies like those of Levy (1994) and Ogaki and Zhang (2001).

Further intuition for the pattern in Figure 1 can be obtained from the first order conditions,

$$\mathrm{E}\left[\left(x(W_0)\cdot(1+r^f+\alpha\cdot r)^{-\gamma^L}+(1-x(W_0))\cdot(1+r^f+\alpha\cdot r)^{-\gamma^H}\right)\cdot r\right]=0, \tag{10}$$

where $x(W_0) = W_0^{\Delta\gamma}/(1+W_0^{\Delta\gamma})$ is a weight function that increases from zero for $W_0 = 0$ to 1 for large values of W_0. Equal weights are implied by $W_0 = 1$. The use of weights in (10) has an obvious effect. For large initial wealth levels, only the first order condition for a standard CRRA optimization problem for known $\gamma = \gamma^L$ plays a role. The opposite holds for low wealth levels. The phenomenon is linked to the use of the transformed probabilities $\bar{G}$ in Theorem 2.1 and can be understood from the different curvatures of the utilities for the two different levels of risk aversion. For high wealth levels, the trade-off between a risky and a safe prospect is dominated by the lowest risk aversion utility function. The curvature of the high-risk aversion (γ^H) utility for high wealth is negligible compared to the curvature of its γ^L counterpart. The converse holds for low levels of wealth, where the curvature of $U(\cdot, \gamma^H)$ dominates that of $U(\cdot, \gamma^L)$. This causes $x(W)$ to go to 0 and the asset allocation (and the first order condition) in this area to be dominated by γ^H.

4. Conclusion

We have shown that uncertainty about risk aversion impacts the relation between wealth and risk aversion, so that the asset allocation implications of traditional utility functions are altered. The relation between wealth and risk-taking depends on the specification of uncertainty. Our example for a power utility maximizer shows that some uncertainty on risk aversion leads to a positive relation between wealth and risk taking. This has implications for analyzing actual risk-taking behavior: preference uncertainty helps to reconcile power utility implied decision making with observed decreasing absolute risk aversion (DRRA) behavior.

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Appendix

Derivation of ARA

Consider the gamble $W = W_0 + e$ vs. $W = W_0 - e$, each with equal probability 0.5. The certainty equivalent $c(e)$ is the dollar amount for which $U(c(e)) = \mathrm{E}[U(W)]$, such that $U(c(0)) = U(W_0)$. We assume $U'(W_0) > 0$. Define $c = c(e)$, $\dot{c} = \dot{c}(e) = \partial c(e)/\partial e$, and $\ddot{c} = \ddot{c}(e) = \partial \dot{c}(e)/\partial e$. Taking first and second order derivatives of $U(c) = \mathrm{E}[U(W)]$ with respect to e and evaluating in $e = 0$, we obtain

$$U'(c)\dot{c} = 0.5U'(W_0) - 0.5U'(W_0) = 0 \qquad \Rightarrow \qquad \dot{c}(0) = 0,$$

and

$$U''(c)\dot{c}^2 + U'(c)\ddot{c} = U''(W_0) \qquad \Rightarrow \qquad \ddot{c}(0) = U''(W_0)/U'(W_0),$$

such that $-\ddot{c}$ is the standard absolute risk aversion (ARA) coefficient.

Proof of Theorem 2.1

First note that $\partial \bar{U}(\gamma)/\partial e|_{e=0} = 0$ and $\partial^2 \bar{U}(\gamma)/\partial e^2|_{e=0} = U''(w)$ for $F = 0.5\delta_{w-e} + 0.5\delta_{w+e}$. Using the definition equation for c, $V(\delta_c, G) = V(0.5\delta_{w-e} + 0.5\delta_{w+e}, G)$, and taking first and second order derivatives with respect to e on both sides and evaluating in $e = 0$, we obtain

$$\int v'(U(c))\ U'(c)\ \dot{c}\ \mathrm{d}G = \int v'(\bar{U}(\gamma))\ \frac{\partial \bar{U}(\gamma)}{\partial e}\ \mathrm{d}G \quad \overset{e=0}{\Longrightarrow} \quad \dot{c}(0, G) = 0,$$

and

$$\int v''(U(c))\ U'(c)^2\ \dot{c}^2\ \mathrm{d}G + \int v'(U(c))\ U''(c)\ \dot{c}^2\ \mathrm{d}G + \int v'(U(c))\ U'(c)\ \ddot{c}\ \mathrm{d}G =$$

$$\int v''(\bar{U}(\gamma))\ \left(\frac{\partial \bar{U}(\gamma)}{\partial e}\right)^2\ \mathrm{d}G + \int v'(\bar{U}(\gamma))\ \frac{\partial^2 \bar{U}(\gamma)}{\partial e^2}\ \mathrm{d}G \quad \overset{e=0}{\Longrightarrow}$$

$$\begin{aligned} \ddot{c}(0, G) \quad &= \quad \frac{\int v'(U(w))\ U''(w)\mathrm{d}G}{\int v'(U(c))\ U'(c)\ \mathrm{d}G} \quad = \quad \int \frac{U''(w)}{U'(w)}\ \frac{v'(U(w))\ U'(w)\ \mathrm{d}G}{\int v'(U(w))\ U'(w)\ \mathrm{d}G} \\ &= \quad \int \frac{U''(w)}{U'(w)}\ \mathrm{d}\bar{G}, \end{aligned}$$

where we used $c(0) = w$. The result for ARA_γ follows similarly.