An Analytic Approach to Credit Risk of Large Corporate Bond and Loan Portfolios

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Proof of Theorem ??: Along the lines of the previous proof, we have to consider
\[ P(C > \pi^* - u_1) = P \left[ \bigcup_{G \in \mathcal{G}} \left\{ \sum_{j \in G} \lambda_j \pi_j \Phi \left( \frac{s - |\hat{R}_j | \hat{v}^\top_j Y}{\sqrt{1 - \hat{R}_j^2}} \right) > \pi^* - u_1 \right\} \right]. \quad (1) \]

The first step is to prove that the events inside the square brackets are disjoint. To see this for \( u_1 \downarrow 0 \), let
\( G_1, G_2 \in \mathcal{G} \) with \( G_1 \neq G_2 \). Consider \( u_1 \) arbitrarily small and a region for \( Y \) such that for \( j = 1, 2 \),
\[ \sum_{j \in G_i} \lambda_j \pi_j \Phi \left( \frac{s - \hat{R}_j | \hat{v}^\top_j Y}{\sqrt{1 - \hat{R}_j^2}} \right) > \pi^* - u_1 \]. \quad (2)

As there is no subset \( G_{s_2} \) of \( G_2 \) such that the inequality (2) is also satisfied for \( G_{s_2} \), there must be a constant \( k > 0 \) such that
\[ \sum_{j \in G_2 \setminus G_1} \lambda_j \pi_j \Phi \left( \frac{s - \hat{R}_j | \hat{v}^\top_j Y}{\sqrt{1 - \hat{R}_j^2}} \right) > k, \]
implying
\[ \sum_{j \in G_2 \setminus G_1} \lambda_j \pi_j \Phi \left( \frac{s - \hat{R}_j | \hat{v}^\top_j Y}{\sqrt{1 - \hat{R}_j^2}} \right) > \pi^* + k - u_1, \]
in the region for \( Y \) considered. This, however, contradicts the definition of \( \pi^* \).

We now have for \( u_1 \downarrow 0 \),
\[ P(C > \pi^* - u_1) = \sum_{G \in \mathcal{G}} P \left[ \sum_{j \in G} \lambda_j \pi_j \Phi \left( \frac{s - \hat{R}_j | \hat{v}^\top_j Y}{\sqrt{1 - \hat{R}_j^2}} \right) > \pi^* - u_1 \right]. \quad (3) \]

Define \( a_j = s/\sqrt{1 - \hat{R}_j^2} \) and \( b_j = \hat{R}_j | \hat{v}_j | \sqrt{1 - \hat{R}_j^2} \), and \( \lambda_j = \lambda_j \pi_j \). Then the probabilities inside the sum in (3) simplify to
\[ P \left[ \sum_{j \in G} \lambda_j \Phi (a_j - b_j^\top Y) > \pi^* - u_1 \right]. \quad (4) \]

Now split \( Y \) in polar coordinates, \( Y = R \theta \), with \( R^2 \) a \( \chi_m^2 \) variate, and \( \theta \) uniform on a hyperglobe. The variates \( R \) and \( \theta \) are independent. Now rewrite (4) as
\[ \int P \left[ \sum_{j \in G} \lambda_j \Phi (a_j - R b_j^\top \theta) > \pi^* - u_1 \mid \theta \right] P(d\theta). \quad (5) \]

Define \( \bar{\Phi}(x) = 1 - \Phi(x) \). Then rewrite (5) as
\[ \int P \left[ \sum_{j \in G} \lambda_j \Phi (a_j - R b_j^\top \theta) < u_1 \mid \theta \right] P(d\theta). \quad (6) \]
Now first consider the probabilities inside the integral. Define $\Theta$ as the set $\theta$’s for which $b_j^\top \theta < 0$ for all $j \in G$. Note that $\Theta$ constitutes the only set of $\theta$’s of interest. For other $\theta$’s, the probability inside the integral equals zero for $u_1 \downarrow 0$.

Next, make a subdivision of $\Theta$ into $\Theta_j$, $j = 1, \ldots, m$, such that we have $|b_j^\top \theta| < |b_i^\top \theta|$ for all $i \neq j$ and $\theta \in \Theta_j$. The $\Theta_j$’s are disjoint. Therefore, we can rewrite (6) as

$$\sum_{j \in G} \int_{\Theta_j} P \left[ \hat{\lambda}_j \Phi \left( a_j - R b_j^\top \theta \right) < u_1 \right] d\theta. \quad (7)$$

Simplify the probability inside the integral as

$$P \left[ R^2 > \left( \Phi^{-1} \left( \frac{u_1}{\hat{\lambda}_j} \right) + a_j \right) \frac{1}{b_j^\top \theta} \right]. \quad (8)$$

From (6.5.4) and (6.5.32) in Abramowitz and Stegun (1970) we have

$$\int_x^\infty e^{-t} t^{a-1} dt = x^{a-1} e^{-x} (1 + O(x^{-1}))$$

for $x \to \infty$. Then from (26.4.19) from Abramowitz and Stegun it follows that for large $x$

$$P(R^2 > x^2) = \frac{(x/2)^{m/2-1} e^{-x^2/2}}{\Gamma(m/2)} (1 + O(x^{-2})).$$

We also have

$$\exp(-\Phi^{-1}(x)^2/2) \approx x \cdot L(x)$$

for $x \uparrow \infty$. Combining all these results and using the independence of $R$ and $\theta$, we can approximate (asymptotically) (8) by

$$\left( \frac{u_1}{\hat{\lambda}_j} \right)^{1/(b_j^\top \theta)^2}. \quad (9)$$

Again combining all results, we have for $u_1 \downarrow 0$

$$P(C > \pi^* - u_1) = \sum_{G \in \mathcal{G}} \sum_{j=1}^m \int_{\Theta_j} \left( \frac{u_1}{\hat{\lambda}_j} \right)^{1/(b_j^\top \theta)^2} P(d\theta). \quad (10)$$

As we are only interested in

$$\alpha = \lim_{u_1 \downarrow 0} \frac{\ln P(C > \pi^* - u_1)}{\ln u_1},$$

if follows from (10) that

$$\alpha = \min \min \text{ess inf}_{\theta \in \Theta_j} (b_j^\top \theta)^{-2} = \min \min \text{ess inf}_{\theta \in \Theta_j} \frac{1 - \hat{R}_j^2}{R_j^2(v_j^\top \theta)^2}, \quad (11)$$

where, to be precise, $\Theta_j = \Theta_j(G)$.

**Remark:** It is only a visual illusion that this result does not seem to nest the result for homogenous $v_j$. Indeed, there is a min over $j$ rather than the max derived in the previous
theorem. However, consider the case of homogenous \( v_j \). In that case, we can simplify to a one-factor model by considering \( v^\top Y \) instead of \( Y \). Note that \( \theta \) can only be 1 or \(-1\) now. Using the proof of the present and the previous theorem, it is easy to see (focus for example on the case \( m = 2 \)) that only one of the \( \Theta_j \)'s will be non-empty, and this non-empty \( \Theta_j \) will contain either only 1 or only \(-1\). The non-empty \( \Theta_j \) is characterized by precisely that \( j \) for which \( |b_j| \) is at its minimum, or \((1 - \hat{R}_j)^2/\hat{R}_j^2 \) is at its maximum, see just above (7). So the minimum over \( j \) in (11) is correct, but one has to bear in mind that several of the \( \Theta_j(G) \)'s may be empty. We can easily accommodate this by defining the ess inf over an empty set to be \(+\infty\).

Note that (11) can be simplified further. Define

\[
\Theta^*(G) = \bigcup_{j \in G} \Theta_j(G),
\]

then the minimum over \( j \) and the infimum over \( \theta \) can be integrated. Note that conditional on a \( \theta \in \Theta^* \), \( j = j(\theta) \) is determined by the smallest \( |b_j^\top \theta| \), i.e., by the maximum \( (b_j^\top \theta)^{-2} \).

Therefore, we have an equivalent expression for (11), namely

\[
\alpha = \min_{G \in \mathcal{G}} \inf_{\theta \in \Theta^*} \max_{j \in G} \frac{1 - \hat{R}_j^2}{\hat{R}_j^2 (v_j^\top \theta)^2}.
\]

(12)

This completes the proof. \( \blacksquare \)