

# Exercise Binomial Tree

## *Principles of Programming in Econometrics*

MSc/PhD Bootcamp  
Vrije Universiteit Amsterdam & Tinbergen Institute  
C.S. Bos, c.s.bos@vu.nl

The exercise below is an alternative exercise, to be solved over the course of the next weeks.

### 1 Setting and model

Your exercise for this exam is to value an option on a stock using the binomial options pricing model (Cox, Ross, and Rubinstein, 1979). In this model, the price of a stock can go up or down by a certain (fixed) percentage, each period. Starting from a price  $S_0$ , after one period either a price a bit higher, or a price a bit lower can be reached, after two periods, three different prices are possible (twice up, up/down= down/up, or twice down), etc. After  $n$  periods the option expires, and depending on the value of the stock  $S_n$  at that period the option pays out, or not.

The question now is: What is the value of the option, if one has  $n$  periods to go, where the probability of an up-tick is known, and the size of the up/down ticks as well?

#### 1.1 The value of the option at expiration

Two option values are considered at expiration:

$$C_n^{\text{Call}}(S, K) = \max(S - K, 0), \quad (1)$$

$$C_n^{\text{Put}}(S, K) = \max(K - S, 0), \quad (2)$$

where  $S_n$  is the spot price at final period  $n$ , and  $K$  is the strike price.

#### 1.2 Step size

Usually, the size of the steps is related to the underlying volatility of the stock, over a year. If  $\sigma^2$  is the yearly variance, and we divide the number of years  $T$  in  $n$  time steps, then  $\Delta t = T/n$  and

$$u = \exp(\sigma\sqrt{\Delta t}),$$
$$d \equiv 1/u = u^{-1}.$$

This step size is used such that a stock moving  $N$  steps, of which  $N_u$  steps up, and  $N - N_u = N_d$  steps down, ends up at

$$S = S_0 \times u^{N_u - N_d} \equiv S_0 \times u^{N_u} \times d^{N_d} = S_0 \times u^{N_u} \times d^{N - N_u}.$$

### 1.3 Probability of an up-tick

At the course of Stochastic Processes for Finance, you might learn that the probability of a move upwards is

$$p = \frac{1}{u - d} (\exp(r\Delta t) - d),$$

and the move downwards occurs with probability  $1 - p$ .

Here we use the yearly risk-free interest rate  $r$ , and assume that the dividend yield is zero, for simplicity.

### 1.4 Pricing an option

The price of an option at period  $j$  is related to the price of an option one period later, according to

$$C_j(S, K, r) = \exp(-r\Delta t) [pC_{j+1}(S \times u, K, r) + (1 - p)C_{j+1}(S/u, K, r)].$$

This is a recursive formula, which can be solved backwards: For the last period, at expiration, the option price  $C_n(S, K)$  is known from either (1) or (2).

Earlier option prices are constructed evaluating this formula backwards.

## 2 The exercise

Take a yearly interest rate of  $r = 0.02$ , and a volatility of  $\sigma = 0.3$ , a strike  $K = 100$ , where you start the stock price at  $S_0 = c \times K$ , with  $c = .75$ . Do this both for a period of  $T = 1$  year, with a value of  $n$  of choice, e.g.  $n = 5$  for starters, and check if you could evaluate for  $n = 10000$  in a decent amount of time.

As a check, Table 1 displays the value of the option according to our version of the program, for the above settings, for different values of  $n$ .

n	0	1	5	10	100	500	1000
call	0.0	0.5573	2.8439	2.7161	2.6020	2.6031	2.6017
put	25.0	23.5772	25.8638	25.7359	25.6218	25.6229	25.6215

Table 1: Value of call/put option, for  $c = 0.75, r = 0.02, \sigma = 0.3, K = 100, T = 1$

Output could be a table where, for different values of  $c$  and  $n$  you compute the resulting call and put option prices, or a graph where you check whether the price you compute for the option indeed converges for increasing values of  $n$ .

If you hand in, then please prepare a very short report (max. 3 pages excluding graphs/tables, probably shorter is already sufficient) where you discuss your findings, possibly the problems you encountered, and indicate if there is something you might want comments on. Hand in code which replicates the results in your report, and make sure that your code is easily adaptable to our choice of a single value of  $n$  and  $c$ , such that we can time it.

## References

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979). “Option pricing: A simplified approach”. In: *Journal of Financial Economics* 7.3, pp. 229–263. DOI: 10.1016/0304-405X(79)90015-1.