A Note on the Normalization of Variables in Dynamic Stochastic General Equilibrium Models

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Abstract

This paper highlights the fact that mean-shifting normalization of variables in DSGE models might influence the accuracy of simulations based on linear approximations. Some illustrations are provided.

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1 Introduction

Dynamic models derived from economic theory often deal with variables defined in measurement units that do not correspond to those of observed data. This reflects the fact that the absolute magnitude of economic data is usually meaningless and that economists are instead interested in relative properties of these variables and the relations between them. However, the linear approximation of functions establishing these relations may not be robust to changes in unit of measurement. Hence, normalization procedures that are mean-shifting can have important effects that should not be ignored. This is especially important when comparing alternative solution methods to DSGE models since normalizing constants may unintentionally enhance the benefits of nonlinear solution methods or obscure the deficiencies of linear ones. The normalization problem is always present since even the abssentist researcher is unwillingly imposing one. In applied work, from the first RBC models of Kydland and Prescott (1982) to today's DSGEs of e.g. Christiano et al. (2005), it seems that their has been widespread lack of attention given to this subject. The literature dealing with nonlinear solution methods for rational expectation models, from Taylor and Uhlig (1990) and Judd (1992) to Aruoba et al. (2006), seems to have also ignored this point when conducting simulation based exercises. The magnitude of these effects is likely to depend on the nature of the functions being approximated, the properties and the size of the dynamic model itself.

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2 Normalization and Linear Approximation

Let $\mathbf{x}_t \in \mathcal{X} \subset \mathbb{R}^k$ denote a k-dimensional vector random variable and $\{\mathbf{x}_t\}_{t=1}^{\infty}$ a stationary ergodic stochastic sequence whose time-invariant conditional density is implicitly defined by the dynamic stochastic model $\mathbf{x}_{t+1} = f(\mathbf{x}_t) + \epsilon_t$ where, for instance, ϵ_t is an iid process that follows some distribution G. Furthermore, suppose that we are interested in linearizing $f : \mathcal{X} \to \mathcal{X}$ about $\mathbf{x}_0$, here taken to be the mean of $\mathbf{x}_t$ (i.e. $\mathbf{x}_0 \equiv E(\mathbf{x}_t)$), although it could also be some unique time-invariant steady-state of a non-stochastic related model.[1] Then assuming that $f \in \mathrm{C}^2(\mathcal{X})$, the space of twice continuously differentiable functions on $\mathcal{X}$, we have by Taylor's theorem that $f(\mathbf{x}_t) = f(\mathbf{x}_0) + Df(\mathbf{x}_0)(\mathbf{x}_t - \mathbf{x}_0) + R_2(\mathbf{x}_t)$ where $R_2(\mathbf{x}_t) = \sum_{|\alpha|=2}(2/\alpha!)\left[\int_0^1 (1-t)D^{\alpha}f(\mathbf{x}_0 + t(\mathbf{x}_t - \mathbf{x}_0))dt\right](\mathbf{x}_t - \mathbf{x}_0)^{\alpha}$, in multi-index notation. The form of the remainder function is of special importance to us since it reveals that R_2 is also a function of $\mathbf{x}_0$. In particular, Taylor's theorem shows that, when linearizing functions, economists should be concerned with (i) the size of the fluctuations of $\mathbf{x}_t$ around $\mathbf{x}_0$, (ii) the curvature of f, i.e. the magnitude of second derivatives $D^{\alpha}f$ on $\mathcal{X}$, and (iii) the implications of normalization procedures that shift $\mathbf{x}_0 \in \mathcal{X}$. While (i) and (ii) are generally well understood, (iii) is often ignored. This paper is precisely devoted to addressing the nature of (iii) and describing the adverse practical consequences of ignoring it. In this respect, we note that the problem must be framed in terms of a stochastic sequence whose variation is in some sense "small", since otherwise approximations errors could easily be unbounded.[2] Fortunately, observation of a sample path $\{\mathbf{x}_t\}_{t=1}^{T} \in \mathcal{X}^T$ gives information to the researcher about the variability of $\mathbf{x}_t$ around $\mathbf{x}_0$, usually summarized by measures such as the sample variance. The researcher is then interested in having a small approximation error, at least within some interval $[\delta\mathbf{x}_0, (1+\delta)\mathbf{x}_0] \subseteq \mathcal{X}$ and the approximation error can be deemed "acceptable" if for some constant $\epsilon > 0$, we have $|R_2(\mathbf{x}_t)| < \epsilon \; \forall (1-\delta)\mathbf{x}_0 \leq \mathbf{x}_t \leq (1+\delta)\mathbf{x}_0$. As shown above, this error might not be invariant w.r.t. $\mathbf{x}_0$. In practice, one simple way of analyzing this is also to look at the linear approximation errors at the boundaries of the interval, which are given by $R_2((1-\delta)\mathbf{x}_0) = f((1-\delta)\mathbf{x}_0) - \sum_{|\alpha|=0}^{|\alpha|=1} \frac{D^{\alpha}f(\mathbf{x}_0)}{\alpha!}((1-\delta)\mathbf{x}_0)^{\alpha}$, and $R_2((1+\delta)\mathbf{x}_0) = f((1+\delta)\mathbf{x}_0) - \sum_{|\alpha|=0}^{|\alpha|=1} \frac{D^{\alpha}f(\mathbf{x}_0)}{\alpha!}((1+\delta)\mathbf{x}_0)^{\alpha}$.

3 Normalization and the Absolute Error of Linear Approximation

For simplicity, suppose that $\dim(\mathbf{x}_t) = 1$, and let $T_f(x_t, x_0)$ denote the linear approximation of f around x_0 evaluated at x_t, i.e. $T_f(x_t, x_0) := f(x_0) + f'(x_0)(x_t - x_0)$. The error of approximation at the bounds $(1-\delta)x_0$ and $(1+\delta)x_0$, can be used to check for error invariance w.r.t. normalization procedures. In particular, $f(x_t) - T_f(x_t, x_0) = g(x_t, x_0)$ is invariant w.r.t. x_0 iff $g(x_t, x_0) = g(x_t)$ (not a function of x_0). For concreteness, we now apply this reasoning to functions that provide simple and intuitive illustrations of the behavior of absolute linear approximation errors. Define the linear approximation errors e_M at the maximum bound by,[3]

$$e_M(x_0, \delta) := f\big(x_0(1+\delta)\big) - \Big[f(x_0) + \frac{\partial f}{\partial x}\Big|_{x=x_0}\big(x_0(1+\delta) - x_0\big)\Big].$$

[1] Setting $\mathbf{x}_0 \equiv E(\mathbf{x}_t)$, $|E(\mathbf{x}_t)| < \infty$ is not only a natural choice but also one that is justified by the objective of minimizing the expected value of the linear approximation error R_2 i.e. $\mathbf{x}_0 = \arg\min_{\mathbf{x}_0 \in \mathcal{X}} E(R_2) \Rightarrow \mathbf{x}_0 = E(\mathbf{x}_t)$.

[2] This can in principle be made precise by requiring, for instance, that for some bounded $\mathcal{X}^* \subseteq \mathcal{X}$ we have $\mathbf{x}_t \in \mathcal{X}^* \; \forall t$ with probability one.

[3] Substituting $(1+\delta)$ by $(1-\delta)$ yields the approximation error $e_m(x_0, \delta)$ at the lower bound.

Clearly, on a function like $f(x_t) = \log(x_t)$, these errors are invariant to normalizations that shift x_0 since,

$$e_M(x_0, \delta) = \log\big(x_0(1+\delta)\big) - \left[\log(x_0) + \frac{1}{x_0}\Big(x_0(1+\delta) - x_0\Big)\right] = -\delta + \log(1+\delta) = e_M(\delta),$$

is not a function of x_0 (which also holds for the minimum bound e_m). The same result can be found by looking at the following form of the Taylor's expansion nth order remainder

$$R_n(x_0, x_t, \xi) \equiv \frac{1}{(n+1)!}\frac{\partial^{n+1} f(\xi)}{\partial \xi^{n+1}}(x_t - x_0)^{n+1} \;\Leftrightarrow\; R_n(x_0, \delta, \delta_\xi) = \frac{1}{(n+1)!}\frac{\partial^{n+1} f(\delta_\xi x_0)}{\partial(\delta_\xi x_0)^{n+1}}(\delta x_0)^{n+1},$$

where $\xi \in (x_0, x_t) \vee (x_t, x_0)$, and on the r.h.s., the remainder is written in terms of a δ deviation from the steady-state x_0, by defining $\delta_\xi \equiv \xi/x_0$ and $\delta = (x_t - x_0)/x_0$. Now, verifying that the remainder is invariant to x_0 for $f(x_t) = \log(x_t)$ is immediate since,

$$\frac{\partial^2 f(\delta_\xi x_0)}{\partial(\delta_\xi x_0)^2} = -\frac{1}{(\delta_\xi x_0)^2} \Rightarrow R_2(x_0, \delta, \delta_\xi) = -\frac{1}{2(\delta_\xi x_0)^2}(\delta x_0)^2 = -\frac{\delta^2}{2\delta_\xi^2} = R_2(\delta, \delta_\xi).$$

However, as we shall now see, this is not the case for those functions typically featured in DSGE models.

3.1 Cobb-Douglas and CRRA Functions

The sensitivity of linear approximation errors w.r.t. normalization procedures is present in commonly used functions such as the Cobb-Douglas $f(k_t, h_t) = k_t^\alpha h_t^{1-\alpha}$ and the CRRA utility function $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$. Indeed, in the case of $u(c_t)$, the approximation errors are not invariant to normalization of its arguments since,

$$R_2(x_0, \delta, \delta_\xi) = -\frac{1}{2}\theta(\delta_\xi x_0)^{-\theta-1}(\delta x_0)^2 = -\frac{1}{2}\theta\delta_\xi^{-\theta-1}\delta^2 x_0^{1-\theta},$$

is invariant to changes in x_0 only when $\theta = 1$ (which takes as a limit case the $\log(x_t)$) and furthermore,

$$0 < \theta < 1 \;\Rightarrow\; \frac{\partial R_2(x_0, \delta, \delta_\xi)}{\partial x_0} > 0\,, \qquad \theta > 1 \;\Rightarrow\; \frac{\partial R_2(x_0, \delta, \delta_\xi)}{\partial x_0} < 0\,.$$

Figure 1 plots the error of approximation of the CRRA utility function for $\theta = 0.5$ (left) and $\theta = 2$ (right), with $c_t = \delta c_0$, $\delta \in [-0.1, 0.1]$. Clearly, the opposite behavior of approximation errors for $\theta < 1$ and $\theta > 1$ makes it hard to produce general advice to economists that is always valid.

Consider now the Cobb-Douglas production function, $f(k_t, h_t) = k_t^\alpha h_t^{1-\alpha}$. The linear approximation error of a (δ_k, δ_h) relative deviation from the expansion point (k_0, h_0) is given by,

$$e_M(k_0, h_0, \delta_k, \delta_h) = k_0^\alpha h_0^{1-\alpha}\Big[(1+\delta_k)^\alpha(1+\delta_h)^{1-\alpha} - \alpha\delta_k - (1-\alpha)\delta_h - 1\Big],$$

which, as Figure 2 reveals, for $k_t = \delta_k k_0$, $\delta_k \in [-0.1, 0.1]$ and $h_t = \delta_h h_0$, $\delta_h \in [-0.1, 0.1]$ is increasing in both dimensions.

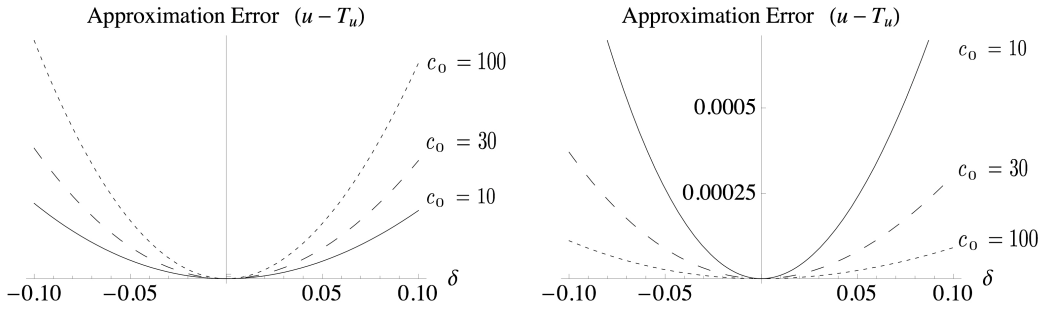

Figure 1: Linear approximation errors of $u(c_t)$, for $\theta = 0.5$ (left) and $\theta = 2$ (right) around c_0.

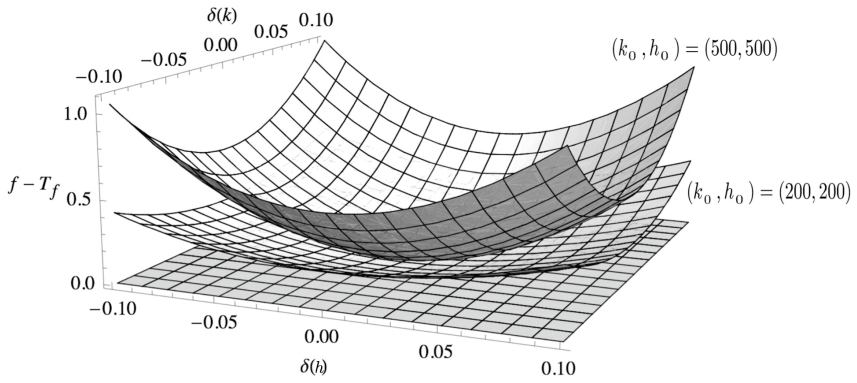

Figure 2: Linear approximation errors of Cobb-Douglas production function $f(k_t, h_t) = k_t^{\alpha} h_t^{1-\alpha}$ about (k_0, h_0).

4 Normalization and the Relative Error of Approximation

One important feature of the Cobb-Douglas and CRRA functions encountered above is their homogeneity. The importance of this property stems from the fact that the relative error of approximation of homogeneous functions is always constant, i.e. even though for any given δ, $u(\delta c_0) - T_u(\delta c_0; c_0)$ and $f(\delta k_0, \delta z_0) - T_f\big((\delta k_0, \delta z_0); (k_0, z_0)\big)$ may change with the choice of c_0 and (k_0, z_0) respectively, we always have that $\big[u(\delta c_0) - T_u(\delta c_0; c_0)\big]/u(c_0)$ and $\big[f(\delta k_0, \delta z_0) - T_f\big((\delta k_0, \delta z_0); (k_0, z_0)\big)\big]/f(k_0, z_0)$ are invariant to normalization. This is important because the relative error of approximation is often more interesting than the absolute one. Consider for instance a production function $y_t = f(x_t)$. Knowing that the error approximation as a fraction of output y_t is invariant to shifts in x_0 might be satisfactory enough in many applications.

Lemma 1 (Homogeneous Function Invariance) *Let $f(\phi x_t) = \phi^{\nu} f(x_t)$ be C^2 in $\mathcal{X}$ and define $e_M^r(x_0, \delta) := e_M(x_0, \delta)/f(x_0)$, $R_2^r(x_0, \delta, \delta_\xi) := R_2(x_0, \delta, \delta_\xi)/f(x_0)$. Then, for $x_t = (1 + \delta)x_0 \in \mathcal{X} \subseteq \mathbb{R}$, $x_0 \in \text{int}(\mathcal{X})$ and*

$\delta_\xi \in (1, 1+\delta) \vee (1+\delta, 1)$, *we have,*

$$e^r_M(x_0, \delta) = (1+\delta)^\nu - (1+\nu\delta) = e^r_M(\delta), \quad \textit{and} \quad R^r_2(x_0, \delta, \delta_\xi) = (\nu^2 - \nu)\frac{\delta^2}{2\delta_\xi^{2-\nu}} = R^r_2(\delta, \delta_\xi),$$

i.e. the relative linear approximation error is invariant to mean-shifting normalization.

As we shall see now, this does not mean, however, that economists should not be concerned with the normalization of variables in dynamic models, even when they have in mind only relative approximation errors.

5 Normalization and Dynamic Models

While homogeneous functions share the interesting invariance property described above, we must be reminded of the fact that difference equations commonly found in structural models, e.g. DSGE models, establish non-homogeneous functional relations between variables at time t and their past. For example, the relative error of a linear approximation of a simple capital accumulation process, $k_{t+1} = (1-\rho)k_t + f(k_t)$ will in general depend on the steady-state of capital k_0 because, even if $f(k_t)$ is homogeneous, $F(k_t) = (1-\rho)k_t + f(k_t)$ is not. Indeed, for $k_{t+1} = (1-\rho)k_t + k_t^\alpha$,

$$R^r_2(x_0, \delta, \delta_c) = \alpha(\alpha-1)\frac{\delta^2}{2\delta_\xi^{2-\alpha}} k_0^\alpha \Big((1-\rho)k_0 + k_0^\alpha\Big)^{-1}.$$

Figure 3 plots the relative linear approximation error of $k_{t+1} = F(k_t) = (1-\rho)k_t + k_t^\alpha$ for $\rho = 0.1$ and $\alpha = 0.5$ around k_0 for $k_t = \delta k_0$, $\delta \in [-0.1, 0.1]$. Note that, in persistent processes, the effect of these may accumulate over time to produce considerable effects on simulated paths.

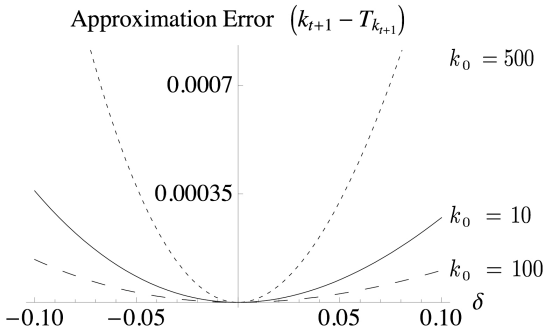

Figure 3: Linear approximation errors of $k_{t+1} = F(k_t) = (1-\rho)k_t + k_t^\alpha$ about k_0.

In general, difference equations in DSGE models are simple composite functions where homogeneous functions are summable on the composition i.e. $F(f_1, .., f_{n_f}) = \sum_{j=1}^{n_f} f_i$ where $f_i(\phi \mathbf{x}_t) = \phi^{\nu_i} f_i(\mathbf{x}_t)$. Lemma 2 below, reveals that, when it comes to approximation error sensitivity to normalization, the different orders of homogeneity of these functions are typically to blame.

Lemma 2 (Composition Invariance) *Let* $F(\mathbf{x}_t) = \sum_{j=1}^{n_f} f_i(\mathbf{x}_t)$ *with* $f_i(\phi\mathbf{x}_t) = \phi^{\nu_i} f_i(\mathbf{x}_t)$. *Then, for* $\mathbf{x}_t = (1+\delta)\mathbf{x}_0 \in \mathcal{X} \subseteq \mathbb{R}^k$, $\mathbf{x}_0 \in \text{int}(\mathcal{X})$ *and* $\delta_\xi \in (1, 1+\delta) \vee \delta_\xi \in (1+\delta, 1)$, *the relative linear approximation error* $F(\delta\mathbf{x}_0) - T_F(\delta\mathbf{x}_0)$ *is invariant to* $\mathbf{x}_0$ *if* $\nu_i = \nu \; \forall \, i$.

In the example above, relative linearization errors are not invariant to normalization because $(1-\rho)k_t$ and k_t^α are homogeneous functions of different degrees. This result is generally applicable to nonlinear dynamic structural models in economics since typically these models are defined by difference equations $F(\mathbf{x}_t) = \sum_{j=1}^{n_f} f_i(\mathbf{x}_t)$ with $f_i(\phi\mathbf{x}_t) = \phi^{\nu_i} f_i(x)$ for which $\exists(i,j) : \nu_i \neq \nu_j$ thus making the composition non-homogeneous.

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