Final exam Advanced Linear Programming, May 27, 13.00-16.00

- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- This exam consists of 4 pages containing 8 questions. Part 1 has 6 questions and part 2 has 2 questions.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.

Good luck, veel succes !

Part 1

(**1**) (1 pt.) Formulate Farkas' Lemma.

Answer.
Theorem. Given $m \times n$ matrix A and $b \in \mathbb{R}^m$, exactly one of the following two alternatives holds:
a) $\exists\, x \geq 0 : \; Ax = b$;
b) $\exists\, y \in \mathbb{R}^m : \; y^T A \geq 0 \wedge y^T b < 0$.

(**2**) (1 pt.) Let A be an $m \times n$ matrix, let C be a $m \times k$ matrix and let $b \in \mathbb{R}^m$. Prove that exactly one of the following holds:
(a) there exist $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$ such that $Ax + Cu \leq b$ and $x \geq 0$;
(b) there exist $y \in \mathbb{R}^m$ such that $y \geq 0$, $y^T A \geq 0$, $y^T C = 0$ and $y^T b < 0$.

Answer. Proof. To turn (a) in the form used in Farkas Lemma as in Exercise 1, we introduce slack variables $s \geq 0$ to induce equality between left- and right-hand side, and we introduce $u^1 \geq 0$ and $u^2 \geq 0$ with $u^1, u^2 \in \mathbb{R}^k$ allowing us to express any $u \in \mathbb{R}^k$ as $u = u^1 - u^2$. This yields the equivalent expression for (a)

(a) there exist $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$ such that $Ax + Cu^1 - Cu^2 + Is = b$ and $x, u^1, u^2, s \geq 0$;
According to Farkas Lemma the corresponding statement for (b) should then be
(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C \geq 0$, $y^T(-C) \geq 0$, $y^T I \geq 0$ and $y^T b < 0$,
which is equivalent to
(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C = 0$, $y \geq 0$ and $y^T b < 0$.

QED

Also correct proofs from scratch have been awarded full points.

(**3**) (1 pt.)
Give the definition of a *vertex* of a polyhedron. Determine all the vertices of

$$\{x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1, \ i = 1, \ldots, n\}.$$

Prove correctness of your answer. You may use any theorems you know as long as you formulate them correctly.

Answer. A *vertex* of a polytope is the only intersection point of the polytope with a hyperplane. A theorem states equivalence of stating that a point is a vertex, a point is a extreme point and a point is a basic feasible solution (bfs). The latter two are easier here for proofs. Define

$$V = \{(x_1, \ldots, x_n) \in P \mid x_i = +1 \text{ or } x_i = -1, i = 1, \ldots, n\}.$$

We need to proof that these 2^n points are indeed all the vertices of P.

A bfs is a point in which n linearly independent constraints defining P hold with equality. P has $2n$ constraints. Clearly, the constraints $x_i \geq -1$ and $x_i \leq 1$ cannot both be met with equality at the same time. Hence, writing the inequality $x_i \geq -1$ as $-e_i^T x \leq -1$ and $x_i \leq 1$ as $e_i^T x \leq 1$, with e_i the i-th unit vector, $i = 1, \ldots, n$, it is clear that

$x \in V$ if and only if x is a bfs (hence a vertex).

QED

Also correct proofs using the definition of extreme point, showing both that $x \in V$ is an extreme point and that every extreme point is in V have been awarded full points.

(**4**) (1 pt.) *Hint: part (b) of this exercise is easier than part (a) and can be made without having part (a) solved correctly.*
Given is the following theorem.

Theorem 0.1 *Let $a_1, \ldots, a_m$ be some vectors in $\mathbb{R}^n$, with $m > n+1$. Suppose that the system of inequalities $a_i^T x \geq b_i$, $i = 1, \ldots, m$, does not have any solutions . Then we can choose $n+1$ of these inequalities, so that the resulting system of inequalities has no solutions.*

(a) Use this theorem to prove Helly's Theorem.

Theorem 0.2 Helly's Theorem. *Let $\mathcal{F}$ be a finite family of polyhedra in $\mathbb{R}^n$ such that every $n+1$ polyhedra in $\mathcal{F}$ have a point in common. Then all polyhedra in $\mathcal{F}$ have a point in common.*

Answer. Proof by contradiction. Suppose that not all polyhedra in $\mathcal{F}$ have a point in common. Think of the inequalities of all polyhedra together as one big system of inequalities. So there is no point that satisfies all of these inequalities. Then the Theorem says that there must be a set of $n+1$ of them that is not satisfied by any point. Take such a subset of $n+1$ inequalities. They belong to at most $n+1$ polyhedra of $\mathcal{F}$. Hence the inequalities of these polyhedra cannot all be simultaneously satisfied. But any set of $n+1$ polyhedra, hence any set of at most $n+1$ polyhedra of $\mathcal{F}$has a point in common. Contradiction.

(b) For $n = 2$, Helly's Theorem asserts that the polyhedra $P_1, P_2, \ldots, P_K$, $(K \geq 3)$ in the plane have a point in common if and only if every three of them have a point in common. Is the result still true with "three" replaced by "two"?

Answer. The answer is NO. It is sufficient to draw an example here, or otherwise define 3 sets that pairwise intersect but do not have a point in the intersection of all three sets.

(**5**) (1 pt.) Consider an uncapacitated network flow problem

$$\begin{aligned} \min \quad & c^T f \\ \text{s.t.} \quad & Af = b \\ & f \geq 0 \end{aligned}$$

and assume that there exists at least one feasible solution. We wish to show that the optimal cost is $-\infty$ if and only if there exists a negative cost directed cycle.

Provide a proof based on the network simplex method.

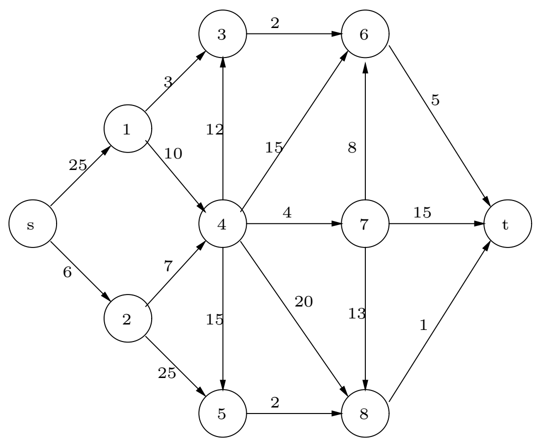
Answer. ($\Leftarrow$). If there exists a negative directed cost cycle then there is a basic cycle C with h^C a vector with all non-negative entries, $Ah^C = 0$, with $c^T h^C < 0$. Clearly $Af = b$

implies $Af + \lambda Ah^C = b$, for all λ. Since there exists no upper bound on the edge capacities we may choose λ arbitrarily large, yielding a infinite cost decrease of $\lambda c^T h^C$.

($\Rightarrow$). If there is no directed negative cost cycle, then in the network simplex algorithm, in every iteration the negative cost cycle found must have a backward arc. This bounds the increase on the augmentation in each iterations. Since the network simplex algorithm terminates in a finite number of steps, the cost will not become $-\infty$.

(6) (1 pt.)

(a) Consider the network given in the figure below. The number at an arc represents the capacity of that arc.



Determine a maximum s–t flow in this network. Show clearly how you compute this cut. Give a short argument why your answer is correct.

(b) Let f be a s-t flow and $\delta^+(U)$ is a s-t cut in any directed graph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$, with capacity function $c : A \to \mathbb{R}_+$, where $\delta^+(U)$ is used to denote the subset of arcs that have their tail in U and their head in $\mathcal{N} \setminus U$. $\delta^-(U)$ is used to denote the subset of arcs that have their tail in $\mathcal{N} \setminus U$ and their head in U. The value of f is the total amount of flow on the arcs with s as their tail, which is equal to the total amount of flow on the arcs with t as their head. Prove that the value of f is equal to the total capacity of $\delta^+(U)$ if and only if the following two statements hold:

$$\begin{aligned} f(a) &= 0 \quad \forall a \in \delta^-(U) \\ f(a) &= c(a) \quad \forall a \in \delta^+(U) \end{aligned}$$

Answer. ($\Leftarrow$). If $f(a) = 0$ for all $a \in \delta^-(U)$ then all the flow going through the cut U from the s-side does not return to this side, hence flows into t. If $\sum_{a \in \delta^+(U)} f(a) = \sum_{a \in \delta^+(U)} c(a)$ then this flow is equal to the cut capacity.

($\Rightarrow$). If the value of f is equal to the total capacity of $\delta^+(U)$ then we know it is a maximum flow, since for any flow and any cut U we have that the value of the flow is at most the size of the cut. The rest of the proof can follow the same arguments as in the proof of the max flow min cut theorem (I don't write this out here).

Part 2

(7) *(2 points) We consider the energy constrained max-flow problem. We are given a directed graph (V, A), where V is the set of nodes and A is the set of arcs. There is a source node $s \in V$ and a sink $t \in V$. Each node i has a battery with capacity E_i. Sending flow on edge (i, j) requires energy from the battery at node i, which amounts e_{ij} per unit flow. The objective is to find the maximal flow s to t, where the flow is required to be integral.*

(a) *Give an integer linear programming formulation for this problem with a* **polynomial** *number of variables (do not use the formulation from part (c)).*
Let the binary decision variable x_{ij} denote the flow on arc (i, j).

$$\begin{array}{rcl} \max & \sum_{(i,t)\in A} x_{ij} & \\ \text{s.t.} & & \\ \sum\limits_{i:(i,j)\in A} x_{ij} & = & \sum\limits_{k:(j,k)\in A} x_{jk} \quad \forall j \in V \setminus \{s,t\} \\ \sum\limits_{j:(i,j)\in A} e_{ij}x_{ij} & \leq & E_i \\ x_{ij} & \in & \mathbb{N} \cup \{0\} \quad \forall (i,j) \in A \end{array}$$

(b) *Show that if we omit the energy contraints, the constraint matrix is totally unimodular.*
When we remove the energy constraint the constraint matrix is given by the following constraints:

$$\sum_{i:(i,j)\in A} x_{ij} - \sum_{k:(j,k)\in A} x_{jk} = 0 \quad \forall j \in V \setminus \{s,t\}$$

Proposition 3.2 form the note on Totally Unimodular matrices states that a matrix is TU if i) $a_{ij} \in -1, 0, 1$ for all i, j, ii) Each column contains at mmost two non-zero coefficients, and iii) there is a partition (M_1, M_2) of the set of rows such each columns containing two-non-zero coefficients satisfies$\sum_{i\in M_1} a_{ij} - \sum_{i\in M_2} a_{ij} = 0$. It is clear that i) is true. A column of a variable x_{sj} has one 1 in row j, column x_{jt} has one -1 in row j. Finally column x_{ij} with $i, j \in V \setminus \{s, t\}$ has a 1 in row j and -1 in row i. This implies ii). Moreover, for each column with two non-zero's we have that teh sum of the elements is 0, so iii) holds with M_1 is the set of all rows.

(c) *It is known that a network flow can be decomposed into a number of $s - t$ paths. An alternative way to formulate the problem is by using these paths. Give an integer linear programming formulation for the problem of* **part (a)** *based on paths.*
Let q be a path from s to t and define parameter $P_{qij} = 1$ if arc (i, j) is on path q and 0 otherwise. Let Q be the set of all paths from s to t. We use binary decision variable x_q which equals 1 if we select path q and 0 otherwise. We obtain the following integer linear programming formulation:

$$
\begin{array}{rcl}
\max & \sum_{q \in Q} x_q & \\
\text{s.t.} & & \\
\sum_{i \in N} P_{qij} e_{ij} x_q & \leq & E_i \quad \forall_{i \in N} \\
x_q & & \text{integer } \forall_{q \in Q}
\end{array}
$$

(d) *Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem. Describe how to solve the pricing problem.*
The problem is solved as follows:

Step 1. Solve the restricted master problem (RMP), i.e. solve the LP-relaxation for a restricted set of variables Q'.

Step 2. Solve the pricing problem, i.e. maximize reduced cost.

Step 3. If the maximum reduced cost is 0 or negative the LP-relaxation has been solved to optimality. Otherwise, i.e. a variable x_q with positive reduced cost is found, add q to Q' and go to Step 1.

Solving the pricing problem
Suppose we have solved the RMP. Let pi_i be the value of dual variable for the energy constraint on node i in the RMP. The reduced cost of variable x_q equals

$$1 - \sum_{(i,j) \in A} P_{qij} e_{ij} \pi_i.$$

The column is defined by the parameters P_{qij} which define the set of arcs that are on the path q. We can rewrite the reduced cost as

$$1 - \sum_{(i,j) \in q} e_{ij} \pi_i$$

. Maximizing the reduced cost, amounts to minimizing $\sum_{(i,j) \in q} e_{ij} \pi_i$. Since q has to be a path from s to t the pricing problem can be solved by solving a shortest path problem, where the cost on the arcs are $e_{ij} \pi_i$. Since, by definition $pi_i \geq 0$ this can be solved by Dijksta's algorithm.

(8) *(2 points)* We consider the minimum spanning tree problem. We are given a graph (V, E), where V is the set of n nodes and E is the set of edges. Edge $e \in E$ has cost c_e. The question is to find a spanning tree with minimal cost, i.e. a subgraph containing all nodes from V which is a tree (graph without cycles).

(a) Give an integer linear programming formulation for this problem in which you include subtour elimination constraints.

(b) Give an integer linear programming formulation for this problem in which you include constraints on cutsets, where for $S \subseteq V$ cutset $\delta(S)$ is defined as $\{(i, j) \in E | i \in S, \, j \notin S\}$.

(c) Let P_{sub} and P_{cut} be the feasible regions of the LP-relaxation of the formulations from part (a) and (b), respectively. Prove that

$$P_{\text{sub}} \subset P_{\text{cut}}$$

and that this is a real inclusion, i.e. the sets are not equal.