Advanced Linear Programming: The Exercises

The answers are sometimes not written out completely.

1.5 a)

$$\begin{array}{ll} \min & c^T x + d^T y \\ \text{s.t.} & Ax + By \leq b \quad y = |x| \end{array} \tag{1}$$

First reformulation, using z smallest number satisfying $x \leq z$ and $-x \leq z$:

$$\begin{array}{ll} \min & c^T x + d^T z \\ \text{s.t.} & Ax + Bz \leq b \\ & x \leq z \\ & -x \leq z. \end{array} \tag{2}$$

Second reformulation, using $x = x^+ - x^-$ and $|x| = x^+ + x^-$:

$$\begin{array}{ll} \min & c^T x^+ - c^T x^- + d^T x^+ + d^T x^- \\ \text{s.t.} & Ax^+ - Ax^- + Bx^+ + Bx^- \leq b \\ & x^+ \geq \underline{0} \\ & x^- \geq \underline{0}. \end{array} \tag{3}$$

1.5 b)
Suppose (1) has feasible solution x (hence $y = |x|$), then choosing $x_i^+ = x_i$ and $x_i^- = 0$ if $x_i \geq 0$ and $x_i^+ = 0$ and $x_i^- = -x_i$ if $x_i \leq 0$ gives a feasible solution of (3). Moreover, the solution values coincide, implying that the optimal value of (3) is at most that of (1).

Suppose (3) has feasible solution x^+, x^- with the property that for all $i = 1, \ldots, n$, $x_i^+ > 0 \Rightarrow x_i^- = 0$ and $x_i^- > 0 \Rightarrow x_i^+ = 0$. Then setting $x_i = x_i^+ - x_i$ and $z = x_i^+ + x_i^-$ gives a feasible solution for (2) with equal value, implying that the optimal value of (2) is at most that of (3). If (3) has a feasible solution that does not have the property then there are variables $x_i^+ > 0$ and $x_i^- > 0$. Setting $\hat{x}_i^+ = x_i^+ - \min\{x_i^+, x_i^-\}$ and $\hat{x}_i^- = x_i^- - \min\{x_i^+, x_i^-\}$ gives $c_i(\hat{x}_i^+ - \hat{x}_i^-) = c_i(x_i^+ - x_i^-)$ and, since $d_i \geq 0$ and the i-th row of B is non-negative, we obtain again a feasible solution and $d_i(\hat{x}_i^+ - \hat{x}_i^-) < d_i(x_i^+ - x_i^-)$. Hence a better feasible solution.

Suppose (2) has feasible solution x, z satisfying for all $i = 1, \ldots, n$ that $z_i = x_i$ or $z_i = -x_i$. Then, setting $y = z$ and keeping x as it is is a feasible solution of (1) with equal value, implying that the optimal value of (1) is at most that of (2). If (2) has a feasible solution that does not have the property then there exist variables $z_i > x_i$ and $z_i > -x_i$. Clearly, since $d_i \geq 0$ and the i-th row of B is non-negative, diminishing the value of z_i yields another feasible solution with better objective value.

1.5 c)
The set $\{x \in \mathbb{R} \mid x - 2|x| \leq -1, |x| \leq 1\}$ is not even connected, let alone convex. Minimizing x gives local optimum $x = 1$, whereas $x = -1$ is the global optimum.

1.15 a)

$$\begin{array}{ll} \max & (9-1.2)x_1 + (8-0.9)x_2 \\ \text{s.t.} & \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90 \\ & \frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array} \tag{4}$$

1.15 b(i))

$$\begin{array}{ll} \max & (9-1.2)x_1 + (8-0.9)x_2 - 7x_3 \\ \text{s.t.} & \frac{1}{4}x_1 + \frac{1}{3}x_2 - x_3 \leq 90 \\ & \frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80 \\ & x_3 + x_4 \leq 50 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{array} \tag{5}$$

1.15 b(ii))
The easiest way is to solve the LP like in a). If the bill for raw material is more than 300, simply add 10% of the bill to the total profit. Also solve the P with discounted prizes for the raw material and add the constraint that the bill without discount exceeds 300: i.e. $1.2x + 0.9y \geq 300$. The best of both solutions is the optimal one.

2.6 a). *Solution by Frank Reinders*
PROOF. $y \in C \Rightarrow y = \sum_{j=1}^{n} \lambda_j A_j$, for $\lambda_j \geq 0$, $j = 1, \ldots, n$. Suppose $|\{j \mid \lambda_j > 0\}| = m + 1$ (more can be dealt with by induction). Number them $\pi(1), \ldots, \pi(m+1)$. Then, $y = \sum_{j=1}^{m+1} \lambda_{\pi(j)} A_{\pi(j)}$.

The equation $\mu_1 A_{\pi(1)} + \ldots + \mu_{m+1} A_{\pi(m+1)} = 0$ has a solution with $|\{j \mid \mu_j \neq 0\}| \geq 1$, since $A_j \in \mathbb{R}^m$, $\forall j$. Hence, $\forall c \in \mathbb{R}$,

$$y = \sum_{j=1}^{m+1} \lambda_{\pi(j)} A_{\pi(j)} = \sum_{j=1}^{m+1} \lambda_{\pi(j)} A_{\pi(j)} - c \sum_{j=1}^{m+1} \mu_j A_{\pi(j)} = \sum_{j=1}^{m+1} (\lambda_{\pi(j)} - c\mu_j) A_{\pi(j)}.$$

Choose c such that $\lambda_{\pi(j)} - c\mu_j \geq 0$, $j = 1, \ldots, m+1$, with equality for at least one of them. (Show how to do this.) □

2.6 b). *Solution by Frank Reinders*

Proof. Similar to the proof of a). Now working with $m+2$ positive λ's and the fact that $A_{\pi(2)}-A_{\pi(1)}, \ldots, A_{\pi(m+2)}-A_{\pi(1)} \in \mathbb{R}^m$ are linearly dependent. Hence non-all-equal 0 values of $\mu_2, \ldots, \mu_{m+2}$ exist with $\sum_{j=2}^{m+2} \mu_j (A_{\pi(j)} - A_{\pi(1)}) = 0$. Choosing $\mu_1 = -\sum_{j=2}^{m+2} \mu_j$ yields $\sum_{j=1}^{m+2} \mu_j = 0$ and $\sum_{j=1}^{m+2} \mu_j A_{\pi(j)} = 0$. Check that again $\forall c \in \mathbb{R}$, we can write $y = \sum_{j=1}^{m+2} (\lambda_{\pi(j)} - c\mu_j) A_{\pi(j)}$, this time also verifying that $\sum_{j=1}^{m+2} (\lambda_{\pi(j)} - c\mu_j) = 1$. Then again, choosing c appropriately will make (at least) one of the $\lambda_{\pi(j)} - c\mu_j$ equal, keeping the others ≥ 0 and their sum equal to 1. □

2.6. *Alternative Solution by Anna Tossenberger*

2.6 a) Let's take $y \in C$, now by the definition of C there exist some $\lambda_1, \ldots, \lambda_n \geq 0$ st. $y = \sum_{i=1}^{n} \lambda_i A_i$ ie. Λ is not empty. Moreover Λ is in its standard form, so we can apply Corollary 2.2 (in B & T), which states that Λ has at least one basic feasible solution, let's denote it with $(\mu_1, \ldots, \mu_n)$. Now according to Theorem 2.4 (in B & T) there are at most m nonzero elements among $\mu_1, \ldots, \mu_n$, since the matrix A with columns A_i has at most m independent rows. So the expression $y = \sum_{i=1}^{n} \mu_i A_i$ satisfies all the criteria of the exercise.

2.6 b) The only difference now is that the coefficient vectors $(\lambda_1, \ldots, \lambda_n)$ must also satisfy $\sum_{i=1}^{n} \lambda_i = 1$. So let's consider the polyhedron
$\Lambda' = \{(\lambda_1, \ldots, \lambda_n) | y = \sum_{i=1}^{n} \lambda_i A_i, \sum_{i=1}^{n} \lambda_i = 1, \lambda_1, \ldots, \lambda_n \geq 0\}$.
This is again in normal form with matrix A', which we can get from the previous matrix A by adding the row $(1, \ldots, 1)$. So now the basic feasible solution of the non-empty polihedron (in standard form) will have at most $m+1$ nonzero components, which finishes the proof.

2.6. *Yet another solution for part b by Hao Hu*

2.6 b) Denote $k = |X|$, and $X = \{x^1, \ldots, x^k\}$. If $y \in$conv.hull(X), then

$$P = \left\{ (\lambda_1, \ldots, \lambda_k) \mid \sum_{i=1}^{k} \lambda_i x^i = y, \ \sum_{i=1}^{k} \lambda_i = 1, \ \lambda_i \geq 0, \ i = 1, \ldots, k \right\}$$

is not empty. Since $\lambda_i \geq 0, \ i = 1, \ldots, k$, P is contained in the positive orthant of $\mathbb{R}^k$. Thus P does not contain a line. By Theorem 2.6 P contains at least one extreme point, say $\bar{\lambda}$. The extreme point is equivalent to a basic feasible solution, requiring that there are k linearly independent constraints active at $\bar{\lambda}$. At most $n+1$ of them can come from the equations $\sum_{i=1}^{k} \lambda_i x^i = y$ and $\sum_{i=1}^{k} \lambda_i = 1$. Hence, the rest, at least $k - (n+1)$, must come from $\bar{\lambda}_i$'s that are 0. Therefore at most $n+1$ entries of $\bar{\lambda}$ are positive.

2.15. *Solution by Panagiotis Saridis*
PROOF. It is easy to check that $x \in L \Rightarrow x \in \{z \in P \mid a_i^T z = b_i,\ i = 1,\dots,n-1\}$. Take $x \in \{z \in P \mid a_i^T z = b_i,\ i =,\dots,n-1\}$. Let A' be the matrix with the $n-1$ rows $a_i,\ i = 1,\dots,n-1$. Then rank$(A') = n-1$, and thus nul.space(A') has dimension 1, hence it can be written as. Clearly, $(x-u),(x-v),(u-v) \in nul.space(A')$ implying that x, u and v lie on one line, ℓ say. u is an extreme point on the intersection $P \cap \ell$, since otherwise it could be expressed as the convex combination of two other points on ℓ in P, contradicting that u is a vertex of P. Same for v. Clearly $\ell \cap P$ can have only two extreme points. This implies that x must be a convex combination of u and v, hence $x \in L$. □

3.7. *Solution essentially by Frank Reinders*
$\Leftarrow$). Suppose x is not optimal $\Rightarrow \exists$ feasible x' with $c^T x' < c^T x \Rightarrow c^T(x'-x) < 0$. Clearly, $A(x'-x) = 0$ and $x'_i \geq x_i$ for all $i \in Z$. Thus $d = x'-x$ is feasible for $Ad = 0,\ d_i \geq 0,\ i \in Z$ and has objective value < 0, a contradiction.

$\Rightarrow$). Suppose there exists a feasible d with $c^T d = \delta < 0$. Then $x + \epsilon d$ is a feasible solution for $\epsilon > 0$ small enough s.t. $x_i + \epsilon d_i \geq 0$ for $i \notin Z$. $c^T(x+\epsilon d) = c^T x + \epsilon\delta < c^T x$, a contradiction.

3.18. *Solution partly by Jeroen Pijnenburg*
a) Not correct

Let N denote the set of non-basic indices. Let d be a feasible direction of the simplex iteration. Then we have,

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i,$$

$d_j = 1$ and $d_i = 0\ \forall i \in N \backslash \{j\}$. Where we have an index $j \in N$ such that the reduced cost $\bar{c}_j$ of the variable x_j is negative.

Then we can compute the difference in cost between the point before and after the simplex iteration by:

$$\begin{aligned} c^T(x+\theta^* d) - c^T x = c^T\theta^* d = \theta^* c^T d = \theta^* \left(c_B^T d_B + \textstyle\sum_{i \in N} c_i d_i\right) \\ = \theta^* \left(\textstyle\sum_{i \in N}(c_i - c_B^T B^{-1} A_i) d_i\right) = \theta^* \left(\textstyle\sum_{i \in N} \bar{c}_i d_i\right) = \theta^* \bar{c}_j < 0 \end{aligned}$$

The last strict inequality follows because $\theta^* > 0$ and $\bar{c}_j < 0$. Therefore an iteration of the simplex algorithm which moves the feasible solution by a positive distance cannot leave the cost unchanged.

b). Correct.
c). Not correct.
d). Not correct. Example with multiple optima.
e). Correct. Simplex finds an optimal bfs. In case of multiple optima, there may be optimal convex combinations of optimal bfs's that have more than m positive components, but these will not be found by Simplex.

3.27 *Solution by Thomas Bosman*
(a). We wish to find a vector $x \in \mathbb{R}^n$ that satisfies $x \geq 0$ (1) and $Ax = 0$ (2), and such that the number of positive components of x is maximized.

We show that this can be accomplished by solving the LP

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n y_i \\ \text{subjectto} & A(z+y) = 0 \\ & y_i \leq 1, \forall i, \\ & z, y \geq 0 \end{array}$$

PROOF. Let $I(x)$ denote the index set of positive elements of x i.e. $I(x) = \{i \mid x_i > 0\}$. If x^* a feasible vector that maximizes $|I(x)|$, then clearly for every $\alpha > 0$ also the vector $\bar{x} = \alpha x^*$ is feasible and maximizes $|I(x)|$. Choosing α large enough we can ensure that $\min_{i \in I\bar{x}}\{\bar{x}_i\} \geq 1$.

Given any vector x setting $y_i = \min\{x_i, 1\} = 1$ and $z_i = x_i - y_i$ clearly gives a feasible solution for the LP. Hence, for $\bar{x}$ this yields the solution $\bar{y}_i = 1$ and $\bar{z}_i = \bar{x}_i - \bar{y}_i$ for all $i \in I(\bar{x})$ and $\bar{z}_i = \bar{y}_i = 0$ for all $i \notin I(\bar{x})$, with value $|I(\bar{x})|$.

Similarly, we can argue that any optimal solution (y, z) to the LP always has $y_i \in \{0, 1\}$ for all i. Setting $x_i = y_i + z_i$ will therefore give a feasible solution with $|I(x)| = \sum y_i$.

Hence, the optimal solutions for both problems have the same value and we can find the maximizing vector by solving the LP. □

(b). The same problem but now with the constraints $x \geq 0$ (1) and $Ax = b$ (3).

The difficulty is now that we cannot be certain that there exist x maximizing the positive elements such that the minimum positive element is at least 1. However, given any x satisfying (1) and (3), with $\min_{i \in I(x)}\{x_i\} = \lambda$ we know that $x_\lambda := \lambda x$, satisfies:

$$Ax_\lambda = \lambda b, \qquad x_\lambda \geq 0$$

and has an equal number of positives as x and no positive element less than 1.

Therefore we can solve the LP :

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} y_i \\ \text{subjectto} & A(z+y) - \lambda b = 0 \\ & y_i \leq 1, \forall i, \\ & z, y \geq 0, \lambda \geq 1 \end{array}$$

The constraint $\lambda \geq 1$ is necessary because otherwise the equality in the LP may reduce the $A(z+y) = 0$.

From an optimal solution (λ, y, z). We can now find a x satisfying (1) and (3) by setting $x_i = \frac{(z_i + y_i)}{\lambda}$. The proof of the equivalence of the optimal solutions of both problems follows the same lines as in (a).

4.26. *Solution by Panagiotis Saridis*

Proof. Notice that the cone $\{x \in \mathbb{R}^n \mid Ax = 0, x \geq 0\}$ contains a vector $x \neq 0$ if and only if it contains a vector x with $\sum_{i=1}^{n} x_i = 1$. Thus we may state **(a)** in the following equivalent form:

(a) There exists a solution to $A'x = b$, $x \geq 0$, with A' the matrix obtained from A by adding an extra last row of all 1-elements, and $b^T = (\underline{0}, 1)$.

Then Farkas Lemma, as stated in Theorem 4.6 of [B&T], that either **(a)** is true or

(b) There exists a vector p' such that $(p')^T A' \geq \underline{0}^T$ and $(p')^T b < 0$.

Let $(p')^T = (p^T, p'_{m+1})$. Then **(b)** in in terms of the original matrix A says that $p^T A + p'_{m+1} \underline{1}^T \geq 0$ and $p'_{m+1} < 0 \Rightarrow p^T A > 0$. □

4.31.

Proof. Consider the LP

$$\begin{array}{ll} \max & \underline{1}^T p, \\ \text{s.t.} & (P - I_n)p = 0, \\ & p \geq 0. \end{array} \tag{6}$$

with dual

$$\begin{array}{ll} \min & \underline{0}^T y, \\ \text{s.t.} & y^T (P - I_n) \geq \underline{1}^T. \end{array} \tag{7}$$

Adding over all constraints of the dual yields $\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} y_i - \sum_{i=1}^{n} y_i \geq n$. Since $\sum_{j=1}^{n} p_{ij} = 1$ the inequality becomes $0 \geq n$, which is clearly impossible. Thus the dual is infeasible. Since $p = 0$ is a solution to (6), this problem must be unbounded, implying the feasible *cone* of (6) contains a solution $p \neq 0$, hence also one with $p^T \underline{1} = 1$. □

4.35.

a. *Solution by Anna Tossenberger*

$$\begin{array}{rl} \max & \underline{0}^T x \\ \text{s.t.} & Ax \leq b \\ & Dx \leq d \end{array}$$

b. *Solution by Thomas Bosman*

PROOF. The dual of the problem in **a** is

$$\begin{array}{rl} \text{minimize} & u^T b + v^T d \\ \text{subjectto} & u^T A + v^T D = \bar{0}^T \\ & u, v \geq 0 \end{array}$$

Since $u = v = \bar{0}$ is a feasible solution with value 0 for the dual and the primal is infeasible, the dual must be unbounded and therefore there exists a solution (u, v) such that:

$$u^T b + v^T d < 0 \iff u^T b < -v^T d$$

Take any such a (u, v) pair.

Equality constraint from (1) yields:

$$u^T A + v^T D = \bar{0}^T \Rightarrow u^T A = -v^T D$$

Since for any point $p \in P$ it holds that $Ap - b \leq 0$ and similarly for points in Q we have $Dq - d \leq 0$, using $u, v \geq 0$ and (2), we get:

$$\begin{array}{rcl} u^T Ap & = & u^T b + u^T (Ap - b) \leq u^T b; \\ u^T Aq & = & -v^T Dq = -v^T d - v^T (Dq - d) \geq -v^T d \end{array}$$

Using $u^T b < -v^T d$, by our choice of u and v, the above inequalities together imply that for the vector $c := u^T A$ we have $c^T p < c^T q$ for all $p \in P$ and $q \in Q$. □

4.39.

PROOF. Let d be defined by $A'd = 0$ with $\text{rank}(A') = n - 1$. Suppose $f, g \in C$

exist such that $d = \lambda f + (1-\lambda)g$ for some $\lambda \in (0,1)$. $f, g \in C \Rightarrow A'f \geq 0$ and $A'g \geq 0$, which together with $A'd = \lambda A'f + (1-\lambda)A'g = 0$ implies that $A'f = 0$ and $A'g = 0$. Since the nulspace of A' has dimension 1, i.e., is a line through $\underline{0}$, d, f and g must be scalar multiples of each other.

Let $I = \{i \mid a_i^T d = 0\}$ and suppose that I contains no set of $n-1$ linearly independent constraints, but it does contain a set of $n-2$ linearly independent constraints. (For I containing less than $n-2$ linearly independent constraints the proof is similar.) Then there exists at least one constraint h independent of the ones in I such that $a_h^T d > 0$, and there exists a vector $y \in C$ such that $a_i^T y = 0$, $\forall i \in I$ and $a_h^T y = 0$, which in fact is an extreme ray of C. But then $f = d - \epsilon y \in C$ and $g = d + \epsilon y \in C$, for $\epsilon > 0$ small enough. □

4.40.
Proof. **(a)**. Suppose P_r is not bounded for some $r \geq 0$. This implies that there exists a point $x \in P_r$ and a direction d such that $\forall \lambda > 0$, $x + \lambda d \in P_r$. $\Rightarrow$ $\sum_{i=1}^n a_i^T x + \lambda \sum_{i=1}^n a_i^T d = r \Rightarrow \sum_{i=1}^n a_i^T d = 0$. Clearly, $d \in C$, i.e., $a_i^T d \geq 0, \forall i$, hence $\sum_{i=1}^n a_i^T d = 0 \Rightarrow a_i^T d = 0, \forall i$.

(b). Clearly, $\sum_{i=1}^n a_i$ is linearly independent of any subset of $n-1$ out of the $\{a_1, \ldots, a_m\}$. Hence, the constraint $\sum_{i=1}^n a_i^T x = r$ together with a set of $n-1$ linearly independent active inequalities from among those that define C define an extreme point of P, while at the same time the $n-1$ active constraints define an extreme ray of C. □

4.44(b).
$\text{lin}(P) = \{x \in \mathbb{R}^2 \mid 2x_1 + x_2 = 0\}$, thus P does not have any extreme points. But neither does it have any extreme rays since the recession cone just consist of the lineality space.

5.9.
Proof.

$$\begin{aligned} F(\lambda b) = \min \quad & c^T x \\ \text{s.t.} \quad & Ax = \lambda b \\ & x \geq 0 \end{aligned}$$

which by strong duality duality is equal to

$$F(\lambda b) = \max \quad \lambda p^T b = \lambda \max p^T b$$
$$\text{s.t.} \quad p^T A x \leq c^T$$

which by strong duality is equal to $\lambda F(b)$. □

6.1.

PROOF. Given an optimal solution to the LP-problem. In the basic feasible optimal solution, at most m variables are positive. Round these up to the nearest integer. Let this give a solution with value K (patterns). Rounding up adds at most m to the value of the LP-solution. However, this may not be a feasible solution, because the number of pieces of width i may exceed the demand b_i. If so, then take any pattern j with $x_j > 0$ that contains width w_i, i.e., with $a_{ij} > 0$. Take one of them and remove one occurrence of width w_i. This leads to the occurrence of a pattern j' which is equal to j but with $a_{ij'} = a_{ij} - 1$, and hence setting $x_{j'} \to x_{j'} + 1$ and $x_j \to x_j - 1$. The total number of patterns does not change. Continuing until of each width exactly b_i is produced gives a solution of value K. □

7.2. *Solution by Frank Reinders*

- x_{ij} := production in month j;
- S_j := stock at the end of year $j-1$;
- a_{ij} := tons of wood from forest i that can be collected in year j;
- u_j := upper bound on tons of wood collected in year j;
- d_j := demand in year j;
- c_j := cost per unit of stock at the end of year $j-1$.

The LP-formulation

$$\begin{array}{lll} \min & \sum_j c_j S_j & \\ \text{s.t.} & S_j = S_{j-1} + \sum_{i=1}^M x_{ij} - d_j, & \forall j \\ & \sum_{i=1}^M x_{ij} \leq u_j, & \forall j \\ & S_j \geq 0, & \forall j \\ & S_0 = 0 & \\ 0 \leq x_{ij} \leq a_{ij} + \sum_{k=1}^{j-1}(a_{ik} - x_{ik}), & \forall i \ \forall j & \end{array}$$

7.14.

PROOF. Take solution f. If $f_{ij} < 0$ then replace the column in A corresponding to (i, j), which by slightly confusing notation is denoted $A_{(i,j)}$, by its negative $-A_{(i,j)}$. This corresponds to reversing the direction of (i, j) in the graph. Doing so for all negative f_{ij} we obtain a matrix A', for which $|f|$, the absolute value of f, satisfies $A'|f| = 0$. Thus we apply Lemma 7.1 to obtain K simple circulations on cycles with forward arcs only. Take any such simple circulation f^{C_k}. For all arcs (i, j) that have not been reversed, define $h_{ij}^{C_k} = f_{ij}^{C_k}$ and for all arcs (i, j) that have been reversed, define $h_{ij}^{C_k} = -f_{ij}^{C_k}$, corresponding to a negative unit of flow on the original arc (i, j). □

7.17a.

PROOF. It is trivially true that the existence of a negative cost cycle with infinite capacities on each of its arcs implies an unbounded optimal solution value.

Now assume we have an optimal flow f with unbounded value. Applying the trick to connect all supply nodes from an auxiliary s and all demand nodes to an auxiliary t with appropriate capacities, and an additional arc (t, s) with capacity B, with B sum of supplies (equal to sum of demands). All costs on these arcs are 0. f is extended to a flow circulation on this network in the obvious way, keeping the same objective value. We call the flow circulation again f.

According to the Flow Decomposition Theorem a finite number of simple circulations $f_1, \ldots, f_k$ exist, corresponding to directed cycles $C_1, \ldots, C_k$, such that f can be written as $f = \sum_{i=1}^{k} a_i f_i$, for positive scalars a_i. Hence there must be at least one of the directed cycles that has negative cost. Suppose that each such negative cost cycle has at least one arc of bounded capacity and let u_i be the minimum capacity of an arc on cycle C_i. This clearly implies that $a_i \leq u_i$. Hence the flow on each arc $(i, j) \in \cup_{i=1}^{k} C_i$ is in $\sum_{i=1}^{k} a_i f_i$ bounded by $\sum_{i=1}^{k} u_i$. This contradicts the fact that f has unbounded value. □

7.17b. For uncapacitated problems , provide a proof based on the network simplex method.

PROOF. In this case trivially the existence of a negative cost cycle (with by definition unbounded capacity) has an unbounded optimal solution value.

Suppose there does not exist any negative directed cost cycle in the network. This implies that in each iteration of the network simplex method any negative cost cycle must have a backward arc, which is bounding the augmentation of the flow. Hence the flow will never become unbounded. □

7.19.
For a maximum s-t-flow to be infinite, there must exist a path from s to t consisting of only uncapacitated arcs. Thus, remove from the graph all other arcs and see if a path from s tot t exists in the remaining graph, which is easily done by exhaustive search in $O(|\mathcal{A}|)$ time.

7.20.
(a). The max flow problem can be written as

$$\begin{array}{ll} \min & \sum_{(s,i)\in\mathcal{A}} f_{si} \\ \text{s.t.} & \sum_{(i,j)\in\mathcal{A}} f_{ij} - \sum_{(j,i)\in\mathcal{A}} f_{ji} = 0,\ \forall i \in \mathcal{N} \setminus \{s,t\} \\ & 0 \le f_{ij} \le u_{ij},\ \forall (i,j) \in \mathcal{A} \end{array}$$

The dual to the max flow problem is

$$\begin{array}{ll} \min & \sum_{(i,j)\in\mathcal{A}} u_{ij} q_{ij} \\ \text{s.t.} & q_{ij} + p_i - p_j \ge 0,\ \forall (i \ne s, j \ne t) \in \mathcal{A} \\ & q_{ij} - p_i \ge 1,\ \forall (s,i) \in \mathcal{A} \\ & q_{ij} + p_j \ge 0,\ \forall (j,t) \in \mathcal{A} \\ q_{ij} \ge 0,\ \forall (i,j) \in \mathcal{A} & \end{array}$$

(b). Take any cut S. Let $A(S) := \{(i,j) \in \mathcal{A} \mid i \in S, j \notin S\}$ and set

$$q_{ij} = \begin{cases} 1 & \forall (i,j) \in A(S) \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the objective value is indeed the size of the cut. We need to show that it can be made into a feasible solution by setting the right values for the p_i's. It is a matter of checking the inequalities to see that given the choice of q above the following is a feasible solution for p:

$$p_i = \begin{cases} -1 & \forall i \in S \\ 0 & \forall i \notin S. \end{cases}$$

By LP-complementary slackness $q_{ij}(u_{ij} - f_{ij}) = 0$. Thus $f_{ij} = u_{ij}\ \forall (i,j) \in A(S)$. Also notice that with the choice of q and p, for any cut S all dual inequalities are satisfied with equality, which gives the freedom to choose any flow satisfying $f_{ij} = u_{ij}\ \forall (i,j) \in A(S)$. Argue this can be made into a feasible flow if only if $A(S)$ corresponds to a minimum cut in the graph.

7.28.
(a). Trivial.

(b). If I am right there is an error here: it should be

$$\bar{\theta}_j = \min\{\theta_j^* - \theta_k^*, c_{kj} + p_j - p_k\}.$$

Since p_j remains the same and $\forall i \in S$, p_i is raised by the same amount θ_k^*, θ_j^* is diminished by θ_k^* and the same index $i \in S$ is minimum. This value must be compared to $c_{kj} + p_j - p_k$.
(c),(d). It could be that not only k becomes labeled, but several other nodes become labeled simultaneously (if k is not the unique $\mathrm{argmin}_{h \notin S}\, \theta_h^*$). In that case we should select the minimum from $\theta_j^* - \theta_k^*$ and $c_{kj} + p_j - p_k$ for all k that become labeled, clearly taking as many basic comparisons as there are newly labeled vertices. Minimizing $\bar{\theta}_j$ over all $j \notin S$ requires $O(n)$ steps. Thus finding one augmenting path takes $O(n^2)$ steps. Implementing this in the primal-dual algorithm, after at most nB augmenting paths we find the optimal solution, yielding overall running time of $O(n^3 B)$.

7.32. *Solution by Frank Reinders*
(a). Let $i \in I$. If there is only one j such that $(i, j) \in \mathcal{A}$ then infinite number of bids of i go to j, whence $j \in J$. Otherwise suppose there is some subset of projects J' with for $j' \in J'$, $(i, j') \in \mathcal{A}$ on which i stops bidding at some point in time. Then infinite bidding on the projects $j \notin J$ with $(i, j) \in \mathcal{A}$ makes them less profitable then some project in J', contradicting the stopping of the bidding on it. Hence $J' = \emptyset$ and for all j with $(i, j) \in \mathcal{A}$, we have $j \in J$.

(b). As we argued before, each project in the set J receives an infinite number of bids. On the other hand, as argued under (a) in the proof of Theorem 7.15, once a project receives a bit it remains assigned to some person. Thus after some point in time the persons not in I stop bidding, implying they have been assigned and there is some project k outside J that never received a bid. But by the infinite number of bids on projects in J from persons in I, these projects will eventually go to a person in I. This implies that eventually all persons in I would receive a project from among J and the bidding would stop, contradicting the infinite bidding.

(c). By (a) and (b) the number of projects that are adjacent to the set of persons I is less than $I|$, whence not every person in I can receive a project.

8.1 *Solution by Frank Reinders.*
First prove the $R^T = R$. Then it is a matter of doing the tedious calculations to find both results.

8.2 *Solution by Frank Reinders.*
I just give a rough sketch of the proof. A crucial observation for the proof is that

$$RD^{1/2}a = \|D^{1/2}a\|e_1 = ((D^{1/2}a)^T D^{1/2}a)^{1/2}e_1 = (a^T Da)^{1/2}e_1$$

implies that

$$e_1 = \frac{RD^{1/2}a}{\sqrt{a^T Da}}.$$

This has the consequence that

$$RD^{1/2}(x-z) - \frac{e_1}{n+1} = RD^{1/2}\left((x-z) - \frac{1}{n+1}\frac{Da}{\sqrt{a^T Da}}\right).$$

Then it is a matter of tedious calculations to prove that

$$\begin{aligned} x \in E' \quad &\Leftrightarrow \quad (x-\bar{z})^T \bar{D}(x-\bar{z}) \leq 1 \\ &\Leftrightarrow \quad \left(RD^{1/2}(x-z) - \frac{e_1}{n+1}\right)^T \left(\frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}e_1 e_1^T\right)\right)^{-1} \left(RD^{1/2}(x-z) - \frac{e_1}{n+1}\right) \\ &\Leftrightarrow \quad T(x) \in E_0'. \end{aligned}$$

10.1.
Choose $M = \min_i\{b_i - f\}$, introduce binary variables y_i, $i = 1, \ldots, m$, and write

$$\begin{aligned} &a_i^T x \geq b_i - (1-y_i)M, \quad \forall i \\ &\textstyle\sum_{i=1}^m y_i \geq k, \\ &y_i \in \{0,1\}, \qquad\qquad \forall i. \end{aligned}$$

10.4 *Solution by Frank Reinders.*
Let A_i be the number of modules of type A to be replaced in year i, and similarly B_i, C_i and D_i. Let E_i denote the number of complete engines to be bought. We use c_i^A to denote the unit costs for module A in year i, and similarly for B, C, D, E, as given in Table 10.2 of the book. And let r_i^A denote the forecasted requirements for module A in year i, and similarly for B, C, D, E, as given in Table 10.1 of the book. Then the LP becomes

$$\begin{array}{lll}
\min & \sum_{X=A,\dots,E} \sum_{i=1,2,3} c_i^X X_i & \\
\text{s.t.} & \sum_{i=1}^{j} X_i + E_i \geq \sum_{i=1}^{j} r_i^X + r_i^E, & i = 1,2,3,\ X = A, B, C, D \\
& E_1 \geq 1 & \\
& E_1 + E_2 + E_3 \geq 3 & \\
& X_i \in \mathbb{Z}, & i = 1,2,3,\ X = A, B, C, D, E
\end{array}$$

10.11 *Solution by Frank Reinders.*
Take $n = m = 2$, then the solution $y_1 = y_2 = \frac{1}{2}$, $x_{11} = x_{22} = 1$, $x_{12} = x_{21} = 0$ is in P_{AFL} but not in P_{FL}.

10.13 *Solution by Frank Reinders.*
PROOF. $P_{\text{mcut}} \subseteq P_{\text{sub}}$: Take any set $S \subseteq \mathcal{N}$. Let $S^C = \mathcal{N} \setminus S$ consist of the nodes $\{v_1, \dots, v_{|S^C|}\}$. Define $C_0 = S$ and $C_i = \{v_i\}$, $i = 1, \dots, |S^C|$. Then for any $x \in P_{\text{mcut}}$ we have

$$\sum_{e \in \delta(C_0, C_1, \dots, C_{|S^C|})} x_e \geq |S^C|,$$

which implies

$$\begin{aligned}
\sum_{e \in E(S)} x_e &= \sum_{e \in \mathcal{E}} x_e - \sum_{e \in \delta(C_0, C_1, \dots, C_{|S^C|})} x_e \\
&\leq n - 1 - |S^C| \\
&= |S| + |S^C| - 1 - |S^C| \\
&= |S| - 1,
\end{aligned}$$

hence, $x \in P_{\text{sub}}$.

$P_{\text{sub}} \subseteq P_{\text{mcut}}$: Take any partition $C_0, C_1, \ldots, C_k$. If $x \in P_{\text{sub}}$ then for any set C_i in the partition we have

$$\sum_{e \in E(C_i)} x_e \leq |C_i| - 1,$$

which implies that

$$\begin{aligned}
\sum_{e \in \delta(C_0, C_1, \ldots, C_k)} x_e &= n - 1 - \sum_{i=0}^{k} \sum_{e \in E(C_i)} x_e \\
&\geq n - 1 - \sum_{i=0}^{k} (|C_i| - 1) \\
&= n - 1 - (n + k + 1) \\
&= k,
\end{aligned}$$

implying $x \in P_{\text{mcut}}$ □

10.14.
Proven in the same way as 10.13 but more straightforward.

11.6.
Let i be the state where a new page starts with i. Let $f(i)$ denote the total attractiveness of the layout for the items $i, i+1 \ldots, n$, given that item i starts at a new page. We wish to find $f(1)$. The following *backward* recursion solves the problem:

$$f(i) = \max_{j=i,i+1,\ldots,n} \{c_{ij} + f(j+1)\},$$

with

$$f(n+1) = 0.$$

11.14.
This problem I explained on the blackboard how to solve and I copy the parts of my notes that treat it here below.

Shortcutting the union of the MST and the PM gives a tour of length at most $Z^{MST} + Z^{PM}$. Notice that the length of a perfect matching on a (even) subset of the nodes has length at most $\frac{1}{2}$ the length of the TSP-tour on the same subset of nodes, hence of length at most $\frac{1}{2}$ the length of the TSP-tour on all the nodes. Therefore,

$$Z^{MST} + Z^{PM} \leq Z^{TSP} + \frac{1}{2} Z^{TSP}$$

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